Math 141 Exam #3 Key (Summer 2017)

1a Recurrence relation: $a_{n+1} = -a_n$, $a_1 = 4$.

1b Explicit formula: $a_n = (-1)^{n+1}4$, $n \geq 1$.

2 We have

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/\ln n} = \lim_{n\to\infty} e^{\ln(1/n)^{1/\ln n}} = \lim_{n\to\infty} e^{\ln(1/n)/\ln n} = \lim_{n\to\infty} e^{-\ln n/\ln n} = e^{-1}.$$

3 In the limit process wherein $n \to \infty$ only the expression for $n > 5000$ is relevant. Thus:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty}\frac{n}{e^n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{1}{e^n} = 0.$$

4 Make sure to reindex to use the usual formula:

$$\sum_{n=1}^{\infty} 3^{-2n} = \sum_{n=1}^{\infty}\left(\frac{1}{9}\right)^n = \sum_{n=0}^{\infty}\frac{1}{9}\left(\frac{1}{9}\right)^n = \frac{1/9}{1-1/9} = \frac{1}{8}.$$

5 Partial fraction decomposition yields

$$\frac{2}{(n+6)(n+7)} = \frac{2}{n+6} - \frac{2}{n+7},$$

and so

$$\begin{aligned} s_k &= \sum_{n=1}^{k}\left(\frac{2}{n+6} - \frac{2}{n+7}\right) \\ &= \left(\tfrac{2}{7} - \tfrac{2}{8}\right) + \left(\tfrac{2}{8} - \tfrac{2}{9}\right) + \left(\tfrac{2}{9} - \tfrac{2}{10}\right) + \cdots + \left(\tfrac{2}{k+5} - \tfrac{2}{k+6}\right) + \left(\tfrac{2}{k+6} - \tfrac{2}{k+7}\right) \\ &= \tfrac{2}{7} - \tfrac{2}{k+7}. \end{aligned}$$

From this we see that

$$\sum_{n=1}^{\infty}\frac{2}{(n+6)(n+7)} = \lim_{n\to\infty} s_k = \lim_{k\to\infty}\left(\frac{2}{7} - \frac{2}{k+7}\right) = \frac{2}{7}.$$

6 We have

$$\int_1^{\infty}\frac{1}{e^x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{e^x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_1^t = -\lim_{t\to\infty}(e^{-t} - e^{-1}) = e^{-1}.$$

Since the integral converges, the series also converges by the Integral Test.

7 We'll use the Limit Comparison Test, comparing the given series with $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{2n-\sqrt[3]{n^2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n - n^{2/3}} = \lim_{n\to\infty} \frac{1}{2 - n^{-1/3}} = \frac{1}{2} \in (0, \infty),$$

and so since $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to diverge, the given series must also diverge.

8 We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{[(n+1)!]^2}{[2(n+1)]!} \cdot \frac{(2n)!}{(n!)^2} \right) = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} \in [0, 1),$$

and so the series converges by the Ratio Test.

9 Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \left| \frac{[-(n+1)]^n}{(n+1)!} \cdot \frac{n!}{(-n)^{n-1}} \right| = \lim_{n\to\infty} \frac{(n+1)^{n-1}}{n^{n-1}} = \lim_{n\to\infty} e^{(n-1)\ln\frac{n+1}{n}} = e > 1,$$

and so the series diverges.

10 We may write the series as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}.$$

The Ratio Test will be inconclusive, so we try the Integral Test. With partial fraction decomposition we find that

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(2x-1)(2x+1)}\,dx &= \lim_{t\to\infty} \frac{1}{2} \int_1^t \left(\frac{1}{2x-1} - \frac{1}{2x+1} \right) dx \\
&= \frac{1}{4} \lim_{t\to\infty} \left[\ln\left(\frac{2x-1}{2x+1} \right) \right]_1^t = \frac{1}{4} \lim_{t\to\infty} \left[\ln\left(\frac{2t-1}{2t+1} \right) - \ln\tfrac{1}{3} \right] \\
&= \tfrac{1}{4}\left(\ln 1 - \ln\tfrac{1}{3} \right) = \tfrac{1}{4} \ln 3.
\end{aligned}$$

Since the integral converges, we conclude that the series also converges.

11 For $n \geq 1$ we have

$$b_n = \frac{1}{\sqrt{n}} > 0,$$

with $b_n \to 0$ as $n \to \infty$. Also,

$$b_{n+1} \leq b_n \quad \Leftrightarrow \quad \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \quad \Leftrightarrow \quad n+1 \geq n \quad \Leftrightarrow \quad 1 \geq 0,$$

and since the last inequality is clearly true, the sequence (b_n) is nonincreasing. Therefore the series converges by the Alternating Series Test. It is conditionally convergent since the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

is a divergent p-series.

12 By the Alternating Series Estimation Theorem, the smallest integer k for which the partial sum

$$\sum_{n=1}^{k-1} \frac{(-1)^n n}{n^4+1}$$

approximates the value of the series with an error of less than 10^{-4} must be the first integer k for which $k/(k^4+1) < 10^{-4}$. Dropping the 1 gives $1/k^3 < 10^{-4}$, and a little trial and error will show that the smallest integer that works is $k = 22$. Thus we have

$$\sum_{n=1}^{21} \frac{(-1)^n n}{n^4+1} \approx \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^4+1}$$

with absolute error less than 10^{-4}.