Math 141 Exam #2 Key (Summer 2017)

1 Factor denominator and make the substitution $u = x+1$:

$$\int_1^3 \frac{2}{(x+1)^2}\,dx = \int_2^4 \frac{2}{u^2}\,du = \left[-\frac{2}{u}\right]_2^4 = \frac{1}{2}.$$

2 Using integration by parts,

$$\int t^2 e^{2t} dt = \frac{1}{2}t^2 e^{2t} - \int te^{2t} dt = \frac{1}{2}t^2e^{2t} - \left(\frac{1}{2}te^{2t} - \frac{1}{2}\int e^{2t} dt\right)$$
$$= \frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + c.$$

3 Using integration by parts,

$$\int x\cos 2x\,dx = \left[\frac{x}{2}\sin 2x\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} \sin 2x\,dx = \frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{2}$$

4 The length of the curve given by a function $f : [a,b] \to \mathbb{R}$ is

$$\mathcal{L} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx,$$

and so, using the Fundamental Theorem of Calculus,

$$\mathcal{L} = \int_e^{e^4} \sqrt{1 + \left(\frac{d}{dx}\int_e^x \sqrt{\ln^2 t - 1}\,dt\right)^2}\,dx = \int_e^{e^4} \sqrt{1 + \left(\sqrt{\ln^2 x - 1}\right)^2}\,dx$$
$$= \int_e^{e^4} \sqrt{\ln^2 x}\,dx = \int_e^{e^4} \ln x\,dx = \big[x\ln x - x\big]_e^{e^4} = 3e^4.$$

5 Set $q = 6\tan\theta$, so $dq = 6\sec^2\theta\,d\theta$. Since $q = 6$ implies $\theta = \pi/4$ and $q = 6\sqrt{3}$ implies $\theta = \pi/3$, we obtain:

$$\int_{\pi/4}^{\pi/3} \frac{36\tan^2\theta \cdot 6\sec^2\theta}{(36\tan^2\theta + 36)^2} d\theta = \int_{\pi/4}^{\pi/3} \frac{\tan^2}{6\sec^2\theta} d\theta = \frac{1}{6}\int_{\pi/4}^{\pi/3} \sin^2\theta\,d\theta = \frac{\pi + 6 - 3\sqrt{3}}{144}.$$

(Note: the formula for $\int \sin^n x\,dx$ on the back of the exam helps.)

6a By partial fractions technique:

$$\frac{12r}{(r-4)^2} = \frac{A}{r-4} + \frac{B}{(r-4)^2} \quad\Rightarrow\quad 12r = A(r-4) + B \quad\Rightarrow\quad A = 12,\ B = 48.$$

So,

$$\int \frac{12r}{(r-4)^2}\,dr = \int\left(\frac{12}{r-4} + \frac{48}{(r-4)^2}\right)dr = 12\ln|r-4| - \frac{48}{r-4} + c.$$

6b We have
$$\frac{x+1}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2},$$
which yields $A = -3/4$, $B = -1/2$, $C = 3/4$. Then
$$\int \frac{x+1}{x^2(x-2)}\,dx = \int \left(\frac{-3/4}{x} - \frac{1/2}{x^2} + \frac{3/4}{x-2} \right) dx = \frac{3}{4} \ln \left| \frac{x-2}{x} \right| + \frac{1}{2x} + c.$$

7a Making the substitution $u = \ln y$,
$$\int_2^\infty \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_2^t \frac{dy}{y \ln y} = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du = \lim_{t\to\infty} \big[\ln|\ln t| - \ln|\ln 2| \big] = \infty.$$
The integral diverges.

7b With the substitution $u = 10 - x$,
$$\begin{aligned}\int_0^{10} \frac{1}{\sqrt[4]{10-x}}\,dx &= \lim_{t\to 10^-} \int_0^t \frac{1}{\sqrt[4]{10-x}}\,dx = \lim_{t\to 10^-} \int_{10}^{10-t} \frac{-1}{\sqrt[4]{u}}\,du \\ &= \lim_{t\to 10^-} \left[-\frac{4}{3} u^{3/4} \right]_{10}^{10-t} = -\frac{4}{3} \lim_{t\to 10^-} \left[(10-t)^{3/4} - 10^{3/4} \right] \\ &= \frac{4(10^{3/4})}{3}.\end{aligned}$$
The integral converges.

8 Area is
$$\begin{aligned}\mathcal{A} &= \int_0^\infty (e^{-bx} - e^{-ax})\,dx = \lim_{t\to\infty} \int_0^t (e^{-bx} - e^{-ax})\,dx = \lim_{t\to\infty} \left[-\frac{1}{b} e^{-bx} + \frac{1}{a} e^{-ax} \right]_0^t \\ &= \lim_{t\to\infty} \left(-\frac{1}{b} e^{-bt} + \frac{1}{a} e^{-at} + \frac{1}{b} - \frac{1}{a} \right) = \frac{1}{b} - \frac{1}{a}.\end{aligned}$$