Math 141 Exam #1 Key (Summer 2017)

1 By the Inverse Function Theorem we have $(f^{-1})'(f(x)) = 1/f'(x)$ for all x in an interval I on which f is one-to-one. Since $f(2) = 6$ and $f'(x) = 6x^2 + 1$, we have

$$(f^{-1})'(6) = (f^{-1})'(f(x) = \frac{1}{f'(2)} = \frac{1}{25}.$$

(Note that $f'(2) = 25 > 0$, so f is strictly increasing on some open interval I containing 2, and thus f is one-to-one on I.)

2a Applying the Chain Rule twice,

$$f'(x) = \frac{1}{2}\left(\ln\sqrt{x}\right)^{-1/2} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4x\sqrt{\ln\sqrt{x}}}.$$

2b Note that $g(x) = e^{\ln(x^{\ln x})} = e^{(\ln x)^2}$, so

$$g'(x) = e^{(\ln x)^2} \cdot 2(\ln x) \cdot \frac{1}{x} = \frac{2x^{\ln x}\ln x}{x}.$$

2c Note that $h(t) = e^{t^2\ln(\tan t)}$, so

$$\begin{aligned} h'(t) &= e^{t^2\ln(\tan t)}\frac{d}{dt}\left[t^2\ln(\tan t)\right] = (\tan t)^{t^2}\left[2t\ln(\tan t) + t^2 \cdot \frac{\sec^2 t}{\tan t}\right] \\ &= t(\tan t)^{t^2}\left[2\ln(\tan t) + t\sec t\csc t\right]. \end{aligned}$$

2d Could use $(\log_a x)' = (\ln x)/(\ln a)$:

$$r'(x) = \frac{1}{\sqrt[3]{8x}\ln 7}(\sqrt[3]{8x})' = \frac{1}{3x\ln 7}.$$

2e By the Chain Rule:

$$\varphi'(z) = -\frac{1}{1+(1/z)^2} \cdot (1/z)' = \frac{1}{1+z^2}.$$

2f By the Chain Rule:

$$y' = 3\sinh^2(e^{-4x}) \cdot \cosh(e^{-4x}) \cdot (-4e^{-4x}) = -12e^{-4x}\sinh^2(e^{-4x})\cosh(e^{-4x}).$$

3 Here $y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \cos x\ln x\right)$. At $x = 1$ we thus have $y' = \sin 1$. So the tangent line has slope $\sin 1$ and contains the point $(1, 1^{\sin 1}) = (1, 1)$. The equation of the line is then

$$y = (\sin 1)x - \sin 1 + 1.$$

4a Let $u = 4e^x + 6$, so $\frac{1}{4}du = e^x\,dx$, and we get

$$\int \frac{e^x}{4e^x+6}dx = \int \frac{1/4}{u}du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(4e^x+6) + c.$$

4b Let $u = \ln(\ln x)$, so $du = \frac{1}{x\ln x}dx$, and the integral becomes

$$\int \frac{1}{u}du = \ln|u| + c = \ln\big|\ln(\ln x)\big| + c.$$

4c Let $u = \sin x$, so $du = \cos x\,dx = \frac{1}{\sec x}dx$. Integral becomes

$$\int e^u du = e^u + c = e^{\sin x} + c.$$

4d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1+\cosh t}dt = \int \frac{1}{1+u}du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

5a Let $u = \ln x$, so $du = \frac{1}{x}dx$. Integral becomes

$$\int_0^{\ln 2e} 3^u du = \left[\frac{3^u}{\ln 3}\right]_0^{\ln 2e} = \frac{3^{\ln 2e} - 1}{\ln 3}.$$

5b Let $u = e^x$, so $du = e^x dx$. Integral becomes

$$\int_{1/\sqrt{3}}^1 \frac{1}{1+u^2}du = \left[\tan^{-1}(u)\right]_{1/\sqrt{3}}^1 = \tan^{-1}(1) - \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

6 First we note that if we take $x^{4/(x+5)}$ to be $(x^4)^{1/(x+5)}$, then evaluation of the expression is always possible for $x < 0$ without straying from the real numbers system. However, for $x < 0$ we must write $\ln(x^4) = 4\ln|x|$, since in any case $x^4 = |x|^4$. Now, with LR indicating use of L'Hôpital's Rule, we have

$$\lim_{x\to-5^-} x^{4/(x+5)} = \lim_{x\to-5^-} \exp\left(\frac{4\ln|x|}{x+5}\right) = \exp\left(\lim_{x\to-5^-}\frac{4\ln|x|}{x+5}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x\to-5^-}\frac{4/x}{1}\right) = e^{-4/5},$$

recalling that $(\ln|x|)' = 1/x$.