Math 141 Exam #4 Key (Summer 2016)

1 Let $f(x) = \sqrt[5]{x}$. The 3rd-order Taylor polynomial centered at 32 for f is

$$\begin{aligned} P_3(x) &= f(32) + f'(32)(x-32) + \frac{f''(32)}{2}(x-32)^2 + \frac{f'''(32)}{6}(x-32)^3 \\ &= 2 + \frac{1}{5}(32)^{-4/5}(x-32) - \frac{2}{25}(32)^{-9/5}(x-32)^2 + \frac{6}{125}(32)^{-14/5}(x-32)^3 \\ &= 2 + \tfrac{1}{80}(x-32) - \tfrac{1}{6400}(x-32)^2 + \tfrac{3}{1{,}024{,}000}(x-32)^3 \end{aligned}$$

and so

$$\sqrt[5]{31} = f(31) \approx P_3(31) = 2 - \tfrac{1}{80} - \tfrac{1}{6400} - \tfrac{3}{1{,}024{,}000} \approx 1.987342773$$

(Note this is quite close to the actual value of 1.987340755....)

2a Clearly the series converges when $x = 0$. Assuming $x \neq 0$, we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3x^{4(n+1)}}{(n+1)!}\cdot\frac{n!}{n^3x^{4n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2x^4}{n^3} = 0$$

for all x, and so by the Ratio Test the series converges on $(-\infty, \infty)$. There are no endpoints to consider here.

2b Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\frac{n^3|x|}{(n+1)^3} = |x|,$$

by the Ratio Test the series converges if $|x| < 1$, which implies $x \in (-1, 1)$.

When $x = 1$ the series becomes

$$\sum \frac{(-1)^{n-1}}{n^3},$$

which converges by the Alternating Series Test. When $x = -1$ the series becomes

$$\sum \frac{(-1)^{2n-1}}{n^3} = -\sum \frac{1}{n^3},$$

which is a convergent p-series. The interval of convergence is therefore $[-1, 1]$.

2c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x-1)^{n+1}}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{(-2)^n(x-1)^n}\right| = \lim_{n\to\infty} 2|x-1|\sqrt[4]{\frac{n}{n+1}} = 2|x-1|,$$

by the Ratio Test the series converges if $2|x-1| < 1$, which implies $x \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

When $x = 1/2$ the series becomes

$$\sum \frac{1}{\sqrt[4]{n}} = \sum \frac{1}{n^{1/4}},$$

which is a divergent p-series. When $x = 3/2$ the series becomes

$$\sum \frac{(-1)^n}{\sqrt[4]{n}},$$

which converges by the Alternating Series Test. The interval of convergence is therefore $\left(\frac{1}{2}, \frac{3}{2}\right]$.

3 From the table provided we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. Differentiate this twice to obtain

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiply both sides by x^2:

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$$

Now replace x with $x/3$:

$$\sum_{n=2}^{\infty} \frac{n(n-1)x^n}{3^n} = \frac{2(x/3)^2}{(1-x/3)^3} = \frac{6x^2}{(3-x)^3}.$$

Thus the function represented by the series is $f(x) = 6x^2/(3-x)^3$.

4a From the table provided,

$$\begin{aligned}
(1+x)^{-2/3} &= \sum_{n=0}^{\infty} \binom{-2/3}{n} x^n = \binom{-2/3}{0} + \binom{-2/3}{1} x + \binom{-2/3}{2} x^2 + \binom{-2/3}{3} x^3 + \cdots \\
&= 1 - \frac{2}{3}x + \frac{(-\frac{2}{3})(-\frac{5}{3})}{2!}x^2 + \frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{3!}x^3 + \cdots \\
&= 1 - \frac{2}{3}x + \frac{5}{9}x^2 - \frac{40}{81}x^3 + \cdots
\end{aligned}$$

4b Set $x = 0.18$ to obtain

$$1.18^{-2/3} \approx 1 - \frac{2}{3}(0.18) + \frac{5}{9}(0.18)^2 - \frac{40}{81}(0.18)^3 = 0.89512.$$

5 From the table provided we have the binomial series

$$\frac{1}{\sqrt{1+x^6}} = (1+x^6)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

for all $x \in (-1,1)$, and so

$$\begin{aligned}
\int_0^{0.5} \frac{1}{\sqrt{1+x^6}}\,dx &= \int_0^{0.5} \left[\sum_{n=0}^{\infty} \binom{-1/2}{n} x^n\right] dx = \sum_{n=0}^{\infty} \left[\int_0^{0.5} \binom{-1/2}{n} x^n\,dx\right] \\
&= \sum_{n=0}^{\infty} \left[\binom{-1/2}{n} \frac{x^{n+1}}{n+1}\right]_0^{0.5} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{(0.5)^{n+1}}{n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-(n-1))}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1}.
\end{aligned}$$

This is an alternating series with $b_0 = \frac{1}{2}$ and

$$b_n = \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1}$$

for $n \geq 1$. We evaluate b_n values: $b_0 = \frac{1}{2}$, $b_1 = \frac{1}{16}$, $b_2 = \frac{1}{64}$, $b_3 = \frac{5}{1024}$, $b_4 = \frac{7}{4096} \approx 1.7 \times 10^{-3}$, and finally $b_5 \approx 6.41 \times 10^{-4}$. Thus b_5 is the first value that is less than 10^{-3}, and by the Alternating Series Estimation Theorem the approximation

$$\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{(n+1)!} \left(\frac{1}{2}\right)^{2n+1} \approx b_0 - b_1 + b_2 - b_3 + b_4 \approx 0.4500$$

will have an absolute error that is less than 10^{-3}.

6 Here

$$x = \frac{3}{t+5} - 2 \quad \Rightarrow \quad t = \frac{3}{x+2} - 5,$$

and so

$$y = t + 1 = \left(\frac{3}{x+2} - 5\right) + 1 = \frac{3}{x+2} - 4.$$

That is,

$$f(x) = \frac{3}{x+2} - 4,$$

and from $t \in [0,10]$ we see that the domain of f (i.e. the attainable values of x) will be the closed interval with endpoints

$$\frac{3}{0+5} - 2 = -\frac{7}{5} \quad \text{and} \quad \frac{3}{10+5} - 2 = -\frac{9}{5}.$$

That is, $\mathrm{Dom}(f) = \left[-\frac{9}{5}, -\frac{7}{5}\right]$.

7 There are many possible parametrizations, but one good one is

$$(x(t), y(t)) = (-1, 0)(1 - t) + (0, 5)t = (t - 1, 5t)$$

for $t \in (-\infty, \infty)$.

8 It helps to multiply by r to get

$$r^2 = 2r\sin\theta + 2r\cos\theta.$$

Then, since $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, we obtain

$$x^2 + y^2 = 2y + 2x.$$

We can improve on this: from $(x^2 - 2x) + (y^2 - 2y) = 0$ we obtain

$$(x - 1)^2 + (y - 1)^2 = 2,$$

which is seen to be the equation of a circle centered at $(1, 1)$ with radius $\sqrt{2}$.

9 The area is

$$\begin{aligned}\mathcal{A} &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left(\sqrt{\cos 2\theta}\right)^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos 2\theta\, d\theta = \frac{1}{4}\Big[\sin 2\theta\Big]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{4}[1 - (-1)] = \frac{1}{2}.\end{aligned}$$