Math 141 Exam #3 Key (Summer 2016)

1a Recurrence relation:

$$a_{n+1} = \frac{(-1)^{n+1}}{|a_n|^{-1}+1}, \quad a_1 = -\frac{1}{2}.$$

1b Explicit formula:

$$a_n = \frac{(-1)^n}{n+1}, \quad n \geq 1.$$

2a We have

$$\lim_{n\to\infty} a_n = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2}{1+1/n}} = \sqrt{2}.$$

2b Using L'Hôpital's Rule where indicated, we find that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \ln\left(\frac{3n+1}{3n-1}\right) = \lim_{n\to\infty} \frac{\ln(3n+1)-\ln(3n-1)}{1/n} \stackrel{\text{LR}}{=} \lim_{n\to\infty} \frac{\frac{3}{3n+1} - \frac{3}{3n-1}}{-\frac{1}{n^2}}$$
$$= \lim_{n\to\infty}\left(\frac{3n^2}{3n-1} - \frac{3n^2}{3n+1}\right) = \lim_{n\to\infty} \frac{6n^2}{9n^2-1} = \frac{2}{3}.$$

3 We have

$$1.7\overline{25} = 1.7 + 0.0\overline{25} = 1.7 + \frac{25}{10^3} + \frac{25}{10^5} + \frac{25}{10^7} + \cdots = 1.7 + 25\sum_{k=0}^{\infty} \frac{1}{10^{2k+3}}$$
$$= 1.7 + \sum_{k=0}^{\infty} \frac{25}{1000}\left(\frac{1}{100}\right)^k = 1.7 + \frac{\frac{25}{1000}}{1-\frac{1}{100}} = 1.7 + \frac{\frac{25}{10}}{100-1}$$
$$= \frac{17}{10} + \frac{5}{198} = \frac{854}{495}.$$

4 Partial fraction decomposition:

$$\frac{1}{(k+p)(k+p+1)} = \frac{A}{k+p} + \frac{B}{k+p+2} \quad \Rightarrow \quad 1 = A(k+p+2) + B(k+p),$$

and so $(A+B)k + (Ap+Bp+2A) = 1$. This gives $A+B=0$ and $Ap+Bp+2A=1$, and finally $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So,

$$s_n = \sum_{k=1}^{n} \frac{1}{(k+p)(k+p+2)} = \sum_{k=1}^{n}\left(\frac{1/2}{k+p} - \frac{1/2}{k+p+2}\right)$$
$$= \frac{1}{2}\left[\left(\frac{1}{p+1} - \frac{1}{p+3}\right) + \left(\frac{1}{p+2} - \frac{1}{p+4}\right) + \left(\frac{1}{p+3} - \frac{1}{p+5}\right) + \cdots + \left(\frac{1}{p+n-1} - \frac{1}{p+n+1}\right) + \left(\frac{1}{p+n} - \frac{1}{p+n+2}\right)\right]$$
$$= \frac{1}{2}\left(\frac{1}{p+1} + \frac{1}{p+2} - \frac{1}{p+n+1} - \frac{1}{p+n+2}\right).$$

From this we see that

$$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+2)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+p)(k+p+2)} = \lim_{n\to\infty} s_n = \frac{1}{2}\left(\frac{1}{p+1} + \frac{1}{p+2}\right).$$

5 We have

$$\int_0^\infty \frac{10}{x^2+9} dx = 10 \lim_{t\to\infty} \int_0^t \frac{1}{x^2+3^2} dx = 10 \lim_{t\to\infty} \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^t$$
$$= \frac{10}{3} \lim_{t\to\infty} \tan^{-1}\left(\frac{t}{3}\right) = \frac{10}{3}\cdot\frac{\pi}{2} = \frac{5\pi}{3}.$$

Since the integral converges, the series also converges by the Integral Test.

6 We'll use the Limit Comparison Test, comparing the given series with $\sum_{n=1}^{\infty} \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{2n-\sqrt[3]{n^2}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-n^{2/3}} = \lim_{n\to\infty} \frac{1}{2-n^{-1/3}} = \frac{1}{2} \in (0,\infty),$$

and so since $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to diverge, the given series must also diverge.

7 We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{[(n+1)!]^3}{[3(n+1)]!}\cdot\frac{(3n)!}{(n!)^3}\right) = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} = \frac{1}{27} \in [0,1),$$

and so the series converges by the Ratio Test.

8 The Ratio Test will turn out to be inconclusive, so we use the Limit Comparison Test and compare with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$, using L'Hôpital's Rule where indicated:

$$\lim_{n\to\infty} \frac{\ln\left(\frac{n+2}{n+1}\right)}{\frac{1}{n}} \overset{\text{LR}}{=} \lim_{n\to\infty} \frac{\frac{1}{n+2}-\frac{1}{n+1}}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+3n+2} = 1 \in (0,\infty).$$

Thus the series diverges by the Limit Comparison Test.

9 For $n \geq 2$ we have

$$b_n = \frac{n-1}{4n^2+9} > 0,$$

with $b_n \to 0$ as $n \to \infty$. Is the sequence $(b_n)_{n=1}^{\infty}$ eventually nonincreasing, meaning $b_{n+1} \leq b_n$ for all sufficiently large n? We have

$$b_{n+1} \leq b_n \quad\Leftrightarrow\quad \frac{n}{4(n+1)^2+9} \leq \frac{n-1}{9n^2+9} \quad\Leftrightarrow\quad n(4n^2+9) \leq (n-1)[4(n+1)^2+9]$$
$$\Leftrightarrow\quad 0 \leq 4n^2-4n-13 \quad\Leftrightarrow\quad 4n(n-1) \geq 13.$$

Clearly $4n(n-1) \geq 13$ holds for all $n \geq 3$, and so $b_{n+1} \leq b_n$ holds for all $n \geq 3$. That is, $(b_n)_{n=1}^{\infty}$ is indeed eventually nonincreasing. Therefore, by the Alternating Series Test, we conclude that the series converges.