Math 141 Exam #2 Key (Summer 2016)

1 Let $u = e^{2x} - 2$, so that $\frac{1}{2}du = e^{2x}\,dx$:

$$\int \frac{e^x}{e^x - 2e^{-x}}dx = \int \frac{e^{2x}}{e^{2x}-2}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|e^{2x}-2| + c.$$

2a Let $u = \ln t$ and $v' = \frac{1}{t^{10}}$, so $u' = \frac{1}{t}$ and $v = -\frac{1}{9t^9}$. Integration by parts gives:

$$\int \frac{\ln t}{t^{10}}dt = -\frac{\ln t}{9t^9} - \int\left(\frac{1}{t}\right)\left(-\frac{1}{9t^9}\right)dt = -\frac{\ln t}{9t^9} + \frac{1}{9}\int \frac{1}{t^{10}}dt = -\frac{\ln t}{9t^9} - \frac{1}{81t^9} + c.$$

2b Let $u = x^2$ and $v' = e^{-3x}$, so $u' = 2x$ and $v = -\frac{1}{3}e^{-3x}$. Integration by parts gives:

$$\int x^2e^{-3x}dx = -\frac{x^2}{3}e^{-3x} + \frac{2}{3}\int xe^{-3x}dx. \tag{1}$$

The integral on the right must itself be handled with integration by parts. Let $u = x$ and $v' = e^{-3x}$, so $u' = 1$ and $v = -\frac{1}{3}e^{-3x}$, and then

$$\int xe^{-3x}dx = -\frac{x}{3}e^{-3x} + \frac{1}{3}\int e^{-3x}dx = -\frac{x}{3}e^{-3x} + \frac{1}{3}\left(-\frac{1}{3}e^{-3x}\right) = -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}.$$

Putting this result into (1) yields

$$\int x^2e^{-3x}dx = -\frac{x^2}{3}e^{-3x} + \frac{2}{3}\left(-\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x}\right) = -\frac{1}{27}(9x^2+6x+2)e^{-3x} + c.$$

3a Let $u = \cos\theta$, so that $\sin\theta\,d\theta = -du$, to get

$$\int \frac{\sin^3\theta}{\cos^2\theta}d\theta = -\int \frac{1-u^2}{u^2}du = \int\left(1 - \frac{1}{u^2}\right)du = u + \frac{1}{u} + c = \cos\theta + \sec\theta + c.$$

3b Let $u = \sec x$, so $du = \sec x\tan x\,dx$, and with the identity $\tan^2 = \sec^2 - 1$ we obtain

$$\begin{aligned}\int \tan^5 x\sec^3 x\,dx &= \int (u^2-1)^2u^2\,du = \int (u^6 - 2u^4 + u^2)\,du \\ &= \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + c = \frac{1}{7}\sec^7 - \frac{2}{5}\sec^5 + \frac{1}{3}\sec^3 + c.\end{aligned}$$

4a Let $s = \tan\theta$, so $ds = \sec^2\theta\,d\theta$. Also $\tan\theta = \frac{1}{2}$ implies $\theta = \tan^{-1}\left(\frac{1}{2}\right)$, and $\tan\theta = 1$ implies $\theta = \frac{\pi}{4}$. Letting $\theta_0 = \tan^{-1}\left(\frac{1}{2}\right)$ for brevity, we have

$$\int_{1/2}^{1} \frac{\sqrt{s^2+1}}{s^2}ds = \int_{\theta_0}^{\frac{\pi}{4}} \frac{\sec^3\theta}{\tan^2\theta}d\theta = \int_{\theta_0}^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^2\theta\sin^2\theta}d\theta.$$

Now let $u = \sin\theta$, so $du = \cos\theta\,d\theta$, so that

$$\int_{\theta_0}^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^2\theta\sin^2\theta}d\theta = \int_{\sin\theta_0}^{\sin\frac{\pi}{4}} \frac{1}{(1-u^2)u^2}du = \int_{1/\sqrt{5}}^{1/\sqrt{2}}\left(\frac{1/2}{1-u} + \frac{1/2}{1+u} + \frac{1}{u^2}\right)du$$

$$= \left[-\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u| - \frac{1}{u}\right]_{1/\sqrt{5}}^{1/\sqrt{2}} = \left[\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| - \frac{1}{u}\right]_{1/\sqrt{5}}^{1/\sqrt{2}}$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \sqrt{2} - \frac{1}{2}\ln\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right) + \sqrt{5}$$
$$= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{5}-1}{\sqrt{5}+1}\right) - \sqrt{2} + \sqrt{5}$$
$$= \ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2} - \sqrt{2} + \sqrt{5}.$$

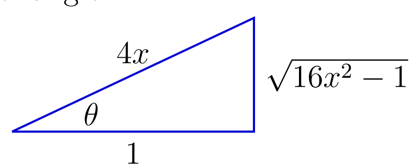
4b Let $x = \frac{1}{4}\sec\theta$ for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = \frac{1}{4}\sec\theta\tan\theta\, d\theta$. Since $x < -\frac{1}{4}$ implies $\sec\theta < -1$ implies $\theta \in (\pi/2, \pi]$, we have $\tan\theta \leq 0$ and hence

$$\sqrt{\tan^2\theta} = |\tan\theta| = -\tan\theta.$$

Now,

$$\int \frac{1}{x^2\sqrt{16x^2-1}}\,dx = 4\int \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\,d\theta = -4\int \cos\theta\, d\theta = -4\sin\theta + c.$$

From $\sec\theta = 4x$ we obtain the triangle



which makes clear that

$$\int \frac{1}{x^2\sqrt{16x^2-1}}\,dx = -4\cdot\frac{\sqrt{16x^2-1}}{4x} + c = -\frac{\sqrt{16x^2-1}}{x} + c.$$

5a We have

$$\frac{12}{(r-4)(r+3)} = \frac{A}{r-4} + \frac{B}{r+3},$$

so

$$12 = A(r+3) + B(r-4) = (A+B)r + (3A-4B),$$

which yields the system of equations

$$\begin{cases} A + B = 0 \\ 3A - 4B = 12 \end{cases} \tag{2}$$

The solution to the system is $(A, B) = \left(\frac{12}{7}, -\frac{12}{7}\right)$, so

$$\int \frac{12}{(r-4)(r+3)}dr = \frac{12}{7}\int \frac{1}{r-4}dr - \frac{12}{7}\int \frac{1}{r+3}dr$$
$$= \frac{12}{7}\ln|r-4| - \frac{12}{7}\ln|r+3| + c = \frac{12}{7}\ln\left|\frac{r-4}{r+3}\right| + c.$$

5b Partial fraction decomposition gives

$$\frac{z+1}{z(z^2+4)} = \frac{\frac{1}{4}}{z} + \frac{-\frac{1}{4}z+1}{z^2+4} = \frac{1}{4z} + \frac{1}{z^2+4} - \frac{z}{4(z^2+4)},$$

and so

$$\begin{aligned}\int \frac{z+1}{z(z^2+4)}\,dz &= \frac{1}{4}\int \frac{1}{z}\,dz + \int \frac{1}{z^2+4}\,dz - \frac{1}{4}\int \frac{z}{z^2+4}\,dz \\ &= \frac{1}{4}\ln|z| + \frac{1}{4}\tan^{-1}\left(\frac{z}{4}\right) - \frac{1}{8}\ln(z^2+4) + c.\end{aligned}$$

6a With partial fraction decomposition we obtain

$$\begin{aligned}\int_{-\infty}^{-2} \frac{2}{t^2-1} dt &= \lim_{a\to-\infty} \int_a^{-2} \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt = \lim_{a\to-\infty} \left[\ln|t-1| - \ln|t+1|\right]_a^{-2} \\ &= \lim_{a\to-\infty} \left(\ln 3 - \ln\left|\frac{a-1}{a+1}\right|\right) = \ln 3 - \ln 1 = \ln 3.\end{aligned}$$

So the integral converges.

6b Note that 1 is not in the domain of the integrand:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \lim_{b\to1^-} \int_0^b \frac{1}{\sqrt{1-x^2}} dx = \lim_{b\to1^-} \left[\sin^{-1} x\right]_0^b = \lim_{b\to1^-} (\sin^{-1} b) = \sin^{-1}(1) = \frac{\pi}{2}.$$

The integral converges.