Math 141 Exam #1 Key (Summer 2016)

1 Note that to say $(-3,5)$ is on the graph of f^{-1} means $f^{-1}(-3)=5$, and this in turn implies that $f(5)=-3$. But in fact

$$f(5)=-5^2+14=-25+14=-11\neq -3,$$

which shows that $(-3,5)$ is *not* on the graph of f^{-1}, and therefore there is no tangent line there.

2a Since $(\ln x)'=x^{-1}$, by the Product Rule we have

$$f'(x)=(x^{-1}\cdot\ln x)'=-x^{-2}\cdot\ln x+x^{-1}\cdot x^{-1}=-\frac{\ln x}{x^2}+\frac{1}{x^2}=\frac{1-\ln x}{x^2}.$$

2b We have

$$\begin{aligned} g'(x)&=\frac{d}{dx}\exp\left[\ln\left(4+\tfrac{2}{x}\right)^{3x}\right]=\frac{d}{dx}\exp\left[3x\ln\left(4+\tfrac{2}{x}\right)\right]=\exp\left[3x\ln\left(4+\tfrac{2}{x}\right)\right]\frac{d}{dx}\left[3x\ln\left(4+\tfrac{2}{x}\right)\right]\\ &=\left(4+\tfrac{2}{x}\right)^{3x}\left[3x\cdot\frac{1}{4+\frac{2}{x}}\cdot\left(-\tfrac{2}{x^2}\right)+3\ln\left(4+\tfrac{2}{x}\right)\right]=\left(4+\frac{2}{x}\right)^{3x}\left[3\ln\left(4+\frac{2}{x}\right)-\frac{3}{2x+1}\right].\end{aligned}$$

2c We have

$$\begin{aligned} h'(t)&=\frac{d}{dt}e^{\ln(\tan t)\sqrt{t}}=\frac{d}{dt}e^{\sqrt{t}\ln(\tan t)}=e^{\sqrt{t}\ln(\tan t)}\left[\sqrt{t}\ln(\tan t)\right]'\\ &=e^{\sqrt{t}\ln(\tan t)}\left[\frac{1}{2\sqrt{t}}\cdot\ln(\tan t)+\sqrt{t}\cdot\frac{1}{\tan t}\cdot\sec^2 t\right]\\ &=(\tan t)^{\sqrt{t}}\left[\frac{\ln(\tan t)}{2\sqrt{t}}+\sqrt{t}\sec t\csc t\right].\end{aligned}$$

2d Using $(\log_a x)'=\frac{1}{x\ln a}$ gives

$$r'(x)=\frac{1}{\sqrt{7x}\ln 2}\cdot\frac{7}{2\sqrt{7x}}=\frac{1}{x\ln 4}.$$

2e $\varphi'(z)=\dfrac{1}{z|\ln z|\sqrt{\ln^2 z-1}}$

2f $y'=4\operatorname{sech}^3(\ln x)\cdot[-\tanh(\ln x)\operatorname{sech}(\ln x)]\cdot\dfrac{1}{x}=-\dfrac{4\operatorname{sech}^4(\ln x)\tanh(\ln x)}{x}.$

3 Note $(x^2)^x=x^{2x}$. Let $f(x)=x^{2x}$, which has domain $(0,\infty)$, and find $x\in(0,\infty)$ for which $f'(x)=0$. That is, find $x>0$ for which

$$(2+2\ln x)x^{2x}=0.$$

This leads to $2+2\ln x=0$, giving $\ln x=-1$, and finally $x=e^{-1}$. Therefore $y=(x^2)^x$ has a horizontal tangent line at the point $(e^{-1},(e^{-2})^{e^{-1}})$.

4a Let $u = 4e^x + 6$, so $\frac{1}{4}du = e^x\,dx$, and we get

$$\int \frac{e^x}{4e^x+6}dx = \int \frac{1/4}{u}du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(4e^x+6) + c.$$

4b We have

$$\int \left(\frac{3}{p-6} - \frac{4}{8p+1}\right) dp = 3\ln|p-6| - \frac{1}{2}\ln|8p+1| + c.$$

4c Let $u = x^8$, so by the Substitution Rule we replace $x^7\,dx$ with $\frac{1}{8}du$ to get

$$\int x^7 8^{x^8} dx = \frac{1}{8}\int 8^u\,du = \frac{1}{8}\cdot\frac{8^u}{\ln 8} + c = \frac{8^{x^8}}{8\ln 8} + c.$$

4d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1+\cosh t}dt = \int \frac{1}{1+u}du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

5a Let $u = x^x$. Now, since $x^x = e^{\ln x^x} = e^{x\ln x}$, we have

$$\frac{du}{dx} = e^{x\ln x}(\ln x + 1) = (1+\ln x)x^x.$$

Thus we formally have $du = (1+\ln x)x^x\,dx$ when we apply the Substitution Method, giving

$$\int_1^2 (1+\ln x)x^x\,dx = \int_1^4 du = 3.$$

5b Let $u = 1/p$, so $-du = \frac{1}{p^2}dp$:

$$\int_{1/3}^{1/2} \frac{10^{1/p}}{p^2}dp = -\int_3^2 10^u\,du = -\left[\frac{10^u}{\ln 10}\right]_3^2 = \frac{1}{\ln 10}(10^3 - 10^2) = \frac{900}{\ln 10}.$$

6 With LR indicating use of L'Hôpital's Rule, we have

$$\lim_{x\to 0}\left(\frac{\sin x}{3x}\right)^{2/x^2} = \lim_{x\to 0}\exp\left(\frac{2}{x^2}\ln\frac{\sin x}{3x}\right) = \exp\left(\frac{\ln\left(\lim\limits_{x\to 0}\frac{\sin x}{3x}\right)}{\lim\limits_{x\to 0}\frac{x^2}{2}}\right)$$

$$\stackrel{\text{LR}}{=} \exp\left(\frac{\ln\left(\lim\limits_{x\to 0}\frac{\cos x}{3}\right)}{\lim\limits_{x\to 0}\frac{x^2}{2}}\right) = \exp\left(\frac{\ln\frac{1}{3}}{\lim\limits_{x\to 0}\frac{x^2}{2}}\right).$$

Since $\ln\frac{1}{3} < 0$ and $\frac{x^2}{2} \to 0^+$ as $x \to 0$, the limit must equal $\exp(-\infty) = 0$.