Math 141 Exam #3 Key (Summer 2014)

1a $a_{n+1} = a_n + 3$, with $a_1 = 4$.

1b $a_n = 3n + 1$ for $n \geq 1$

2 $M_n = 50(0.7)^n$ for $n \geq 0$

3 We have

$$\lim_{n\to\infty} \frac{4n - 5n^6}{11n^6 - 9} = \lim_{n\to\infty} \frac{4n^{-5} - 5}{11 - 9n^{-6}} = \frac{4(0) - 5}{11 - 9(0)} = -\frac{5}{11}.$$

4 Get this into the form $\sum_{n=0}^{\infty} ar^n$:

$$\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n 3^{4-n} = \sum_{n=0}^{\infty} 81\left(\frac{1}{18}\right)^n = \frac{81}{1 - \frac{1}{18}} = \frac{1458}{17}.$$

5 By partial fraction decomposition,

$$\frac{2}{(n+2)(n+4)} = \frac{1}{n+2} - \frac{1}{n+4},$$

so for each $n \geq 1$ we have

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{2}{(k+2)(k+4)} = \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k+4}\right) \\ &= \left(\tfrac{1}{3} - \tfrac{1}{5}\right) + \left(\tfrac{1}{4} - \tfrac{1}{6}\right) + \left(\tfrac{1}{5} - \tfrac{1}{7}\right) + \left(\tfrac{1}{6} - \tfrac{1}{8}\right) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+2}\right) + \left(\tfrac{1}{n+1} - \tfrac{1}{n+3}\right) + \left(\tfrac{1}{n+2} - \tfrac{1}{n+4}\right) \\ &= \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{n+3} - \tfrac{1}{n+4}, \end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{2}{(n+2)(n+4)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{n+3} - \tfrac{1}{n+4}\right) = \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{7}{12}.$$

6a Making the substitution $u = \ln x$,

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2} du = \lim_{b\to\infty} \left[\frac{1}{\ln 2} - \frac{1}{\ln b}\right] = \frac{1}{\ln 2}.$$

So the integral converges, and therefore the series converges by the Integral Test.

6b Since

$$\lim_{n\to\infty} \frac{1}{n+1000} = 0,$$

the Divergence Test is inconclusive.

6c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{[(n+1)!]^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1,$$

the series converges by the Ratio Test.

6d Since

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{2n^2}\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2n} = e^{-2} \approx 0.1353 < 1,$$

the series converges by the Root Test.

6e For each $n \geq 1$ we have

$$0 \leq \frac{1}{n^2+8} \leq \frac{1}{n^2},$$

and since $\sum_{n=1}^{\infty} 1/n^2$ is a convergent p-series, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+8}$$

converges by the Direct Comparison Test.

6f We could use the Limit Comparison Test:

$$\lim_{n\to\infty}\left(\frac{\sqrt[3]{n^2+1}}{\sqrt{n^3+2}} \div \frac{1}{n}\right) = \lim_{n\to\infty}\frac{n^{5/3}\sqrt{1+1/n^2}}{n^{3/2}\sqrt[3]{1+2/n^3}} = \lim_{n\to\infty}\frac{n^{1/6}\sqrt{1+1/n^2}}{\sqrt[3]{1+2/n^3}} = \infty,$$

since

$$\lim_{n\to\infty} n^{1/6} = \infty \quad \text{and} \quad \lim_{n\to\infty}\frac{\sqrt{1+1/n^2}}{\sqrt[3]{1+2/n^3}} = \frac{\sqrt{1+0}}{\sqrt[3]{1+0}} = 1.$$

Now, $\sum \frac{1}{n}$ is a divergent p-series, and so by the Limit Comparison Test we conclude that the given series diverges.

7a Observe that, using L'Hôpital's Rule,

$$\lim_{n\to\infty}\frac{n}{\ln n} = \lim_{n\to\infty}\frac{1}{1/n} = \lim_{n\to\infty} n = \infty,$$

so that

$$\lim_{n\to\infty}(-1)^n\frac{n}{\ln n} \neq 0$$

and the Divergence Test implies that the series diverges.

7b The sequence

$$\left(\frac{\sqrt{n}}{n+6}\right)_{n=1}^{\infty}$$

is monotone decreasing, and moreover

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+6} = \lim_{n\to\infty} \frac{n^{-1/2}}{1+6/n} = 0.$$

Therefore the series converges by the Alternating Series Test.