Math 141 Exam #1 Key (Summer 2014)

1 We have $f'(x) = 5x^4+6x^2+8$, so it is clear that $f'(x) > 0$ for all $x \in (-\infty, \infty)$, which implies that f is an increasing function and therefore is one-to-one. Also f is everywhere differentiable. Given these considerations, a theorem states that if $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$. Now, we are given $f(-2) = -76$, and hence

$$(f^{-1})'(-76) = \frac{1}{f'(-2)} = \frac{1}{5(-2)^4 + 6(-2)^2 + 8} = \frac{1}{112}.$$

2 Let f_1 denote the function f restricted to the interval $[0, \infty)$. Then f_1 is one-to-one, and to find the inverse we solve $y = 2/(x^2 + 2)$ for x:

$$y = \frac{2}{x^2+2} \quad \Leftrightarrow \quad x^2y + 2y = 2 \quad \Leftrightarrow \quad x^2 = \frac{2-2y}{y} \quad \Leftrightarrow \quad x = \sqrt{\frac{2-2y}{y}}.$$

Thus we have

$$f_1^{-1}(y) = \sqrt{\frac{2-2y}{y}}$$

as one local inverse for f, with domain $\mathrm{Dom}(f_1^{-1}) = \mathrm{Ran}(f_1) = (0, 1]$.

Now let f_2 denote the function f restricted to the interval $(-\infty, 0]$. Then f_2 is one-to-one, and to find the inverse we solve $y = 2/(x^2 + 2)$ for x:

$$y = \frac{2}{x^2+2} \quad \Leftrightarrow \quad x^2y + 2y = 2 \quad \Leftrightarrow \quad x^2 = \frac{2-2y}{y} \quad \Leftrightarrow \quad x = -\sqrt{\frac{2-2y}{y}},$$

where $\sqrt{x^2} = |x| = -x$ since $x \le 0$. Thus we have

$$f_2^{-1}(y) = -\sqrt{\frac{2-2y}{y}}$$

is another local inverse for f, with domain $\mathrm{Dom}(f_2^{-1}) = \mathrm{Ran}(f_2) = (0, 1]$.

Since the domains of the two restrictions f_1 and f_2 cover the entire domain $(-\infty, \infty)$ of f, there are no other local inverses to find.

3a $f'(x) = -\dfrac{7e^{-7x}}{e^{-7x}+1}$

3b $\mathrm{Dom}(g) = (0, \infty)$, and for all $x > 0$ we have

$$g(x) = x^{\ln(x^5)} = \exp\left(\ln\left(x^{\ln(x^5)}\right)\right) = \exp\big(\ln(x^5)\ln(x)\big) = \exp\big(5\ln^2(x)\big),$$

and thus

$$g'(x) = \exp\big(5\ln^2(x)\big) \cdot \big(5\ln^2(x)\big)' = x^{\ln(x^5)} \cdot \frac{10\ln(x)}{x} = \frac{10x^{\ln(x^5)}\ln(x)}{x}.$$

3c For x such that $\sin x > 0$ we have

$$h(x) = (\sin x)^{\tan x} = \exp\big(\ln((\sin x)^{\tan x})\big) = \exp(\tan x \cdot \ln(\sin x)),$$

and thus

$$\begin{aligned} h'(x) &= \exp(\tan x \cdot \ln(\sin x))\cdot(\tan x \cdot \ln(\sin x))' \\ &= \exp(\tan x \cdot \ln(\sin x))\cdot\left(\tan x \cdot \frac{\cos x}{\sin x} + \sec^2 x \cdot \ln(\sin x)\right) \\ &= (\sin x)^{\tan x}\left(1 + \ln(\sin x)^{\sec^2 x}\right) \end{aligned}$$

3d $k'(x) = \dfrac{7}{(4 - x^5)\ln(3)} \cdot (4 - x^5)' = -\dfrac{35x^4}{(4 - x^5)\ln(3)}$

3e $\ell'(x) = \dfrac{1}{e^{-2x}\sqrt{(e^{-2x})^2 - 1}} \cdot (e^{-2x})' = \dfrac{-2e^{-2x}}{e^{-2x}\sqrt{e^{-4x} - 1}} = -\dfrac{2}{\sqrt{e^{-4x} - 1}}$

3f $p'(x) = -\dfrac{1}{\sqrt{1 - (\sqrt{x}\,)^2}} \cdot (\sqrt{x}\,)' = -\dfrac{1}{\sqrt{1 - x}} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x - x^2}}$

4a $\displaystyle\int (3e^{-8x} - 8e^{11x})dx = -\frac{3}{8}e^{-8x} - \frac{8}{11}e^{11x} + c$

4b $\displaystyle\int \frac{9}{4 - 9y}dy = -\ln|4 - 9y| + c$

4c Let $u = x^4$, so by the Substitution Rule we replace $x^3\,dx$ with $\frac{1}{4}du$ to get

$$\int x^3 10^{x^4} dx = \frac{1}{4}\int 10^u\,du = \frac{1}{4}\cdot\frac{10^u}{\ln 10} + c = \frac{10^{x^4}}{4\ln 10} + c.$$

5a Let $u = \ln(x)$, so when $x = 1$ we have $u = \ln(1) = 0$, and when $x = 3e$ we have $u = \ln(3e)$. Now, by the Substitution Rule we replace $\frac{1}{x}dx$ with du to get

$$\int_0^{\ln(3e)} \frac{e^u}{2}\,du = \left[\frac{1}{2}e^u\right]_0^{\ln(3e)} = \frac{1}{2}(e^{\ln(3e)} - e^0) = \frac{3e - 1}{2}.$$

5b We have

$$5\int_2^{2\sqrt{3}} \frac{1}{z^2+2^2}dz = 5\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_2^{2\sqrt{3}} = \frac{5}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right] = \frac{5}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{5\pi}{24}.$$

6 For all $x > 0$ we have

$$\left(\frac{2}{x}\right)^{5/x} = \exp\left[\ln\left(\frac{2}{x}\right)^{5/x}\right] = \exp\left[\frac{5}{x}\ln\left(\frac{2}{x}\right)\right] = \exp\left(\frac{5\ln(2/x)}{x}\right).$$

The functions $f(x) = 5\ln(2/x)$ and $g(x) = x$ are differentiable on $(0,\infty)$, and $g'(x) = 1 \neq 0$ for all $x \in (0,\infty)$. Since $g(x) \to \infty$ as $x \to \infty$, and

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{-5/x}{1} = 0,$$

by L'Hôpital's Rule we obtain

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{5\ln(2/x)}{x} = 0$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to\infty}\left(\frac{2}{x}\right)^{5/x} = \lim_{x\to\infty}\exp\left(\frac{5\ln(2/x)}{x}\right) = \exp\left(\lim_{x\to\infty}\frac{5\ln(2/x)}{x}\right) = \exp(0) = 1.$$