Math 141 Exam #3 Key (Summer 2013)

1a We have

$$\lim_{n\to\infty} \frac{5n^8}{\sqrt{36n^{16}-10n^{10}}} = \lim_{n\to\infty} \frac{5n^8}{n^8\sqrt{36-10/n^6}} = \lim_{n\to\infty} \frac{5}{\sqrt{36-10/n^6}} = \frac{5}{\sqrt{36}} = \frac{5}{6}.$$

1b First we evaluate

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{n} &= \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp(\ln n^{1/n}) = \exp\left(\lim_{n\to\infty} \ln n^{1/n}\right) \\ &= \exp\left(\lim_{n\to\infty} \frac{\ln n}{n}\right) \stackrel{LR}{=} \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) = \exp(0) = 1,\end{aligned}$$

where "LR" indicates an application of L'Hôpital's Rule.

Now, consider the subsequence of $\{a_n\}_{n=1}^{\infty}$ that consists of the even-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k$ for $k \geq 1$. Then, using the fact that $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{n_k} n_k^{1/n_k} = \lim_{k\to\infty} (-1)^{2k}(2k)^{1/(2k)} = \lim_{k\to\infty} (2k)^{1/(2k)} = 1.$$

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k-1$ for $k \geq 1$. Then we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{2k-1}(2k-1)^{1/(2k-1)} = \lim_{k\to\infty} \left[-(2k-1)^{1/(2k-1)}\right] = -1.$$

Since $\{a_n\}$ has two subsequences with different limits, the sequence $\{a_n\}$ itself cannot converge. That is, $\{a_n\}$ diverges.

2 Starting by reindexing, we have

$$\sum_{k=2}^{\infty} \frac{3}{(-2)^k} = \sum_{k=0}^{\infty} \frac{3}{(-2)^{k+2}} = \sum_{k=0}^{\infty} \frac{3}{4}\left(-\frac{1}{2}\right)^k = \frac{3/4}{1-(-1/2)} = \frac{1}{2}.$$

3 Partial fraction decomposition gives

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2},$$

so series becomes

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

Now,

$$\begin{aligned} s_n &= \sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right) \\ &= \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\ &= \frac{1}{2}-\frac{1}{n+2}, \end{aligned}$$

so

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)=\frac{1}{2}.$$

4a Since

$$\lim_{k\to\infty}\frac{k}{\sqrt{k^2+25}} = 1 \neq 0,$$

the series diverges by the Divergence Test.

4b Letting $u=-2x^2$, we have

$$\begin{aligned} \int_1^{\infty} xe^{-2x^2}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-2x^2}\,dx = \lim_{b\to\infty}\int_{-2}^{-2b^2} -\frac{1}{4}e^u\,du = \lim_{b\to\infty} -\frac{1}{4}[e^u]_{-2}^{-2b^2} \\ &= \lim_{b\to\infty} -\frac{1}{4}\left(e^{-2b^2}-e^{-2}\right) = -\frac{1}{4}(0-e^{-2}) = \frac{e^{-2}}{4}, \end{aligned}$$

so the integral

$$\int_1^{\infty} xe^{-2x^2}\,dx$$

converges, and therefore the series converges by the Integral Test.

4c Since

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{[(k+1)!]^2}{[2(k+1)]!}\cdot\frac{(2k)!}{(k!)^2}\right| = \lim_{k\to\infty}\frac{(k+1)(k+1)}{(2k+1)(2k+2)} = \lim_{k\to\infty}\frac{k+1}{4k+2} = \frac{1}{4} < 1,$$

the series converges by the Ratio Test.

4d Since

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\frac{k^2}{2^k}} = \lim_{k\to\infty}\frac{k^{2/k}}{2} = \frac{1}{2} < 1,$$

the series converges by the Root Test.

4e For each $k \geq 1$ we have

$$0 \leq \frac{\sin^2 k}{k\sqrt{k}} \leq \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}},$$

and since $\sum_{k=1}^{\infty} k^{-3/2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^{\infty} \frac{\sin^2 k}{k\sqrt{k}}$$

converges by the Direct Comparison Test.

4f For each $k \geq 1$ we have

$$0 \leq \frac{k^7}{k^9 + 3} \leq \frac{k^7}{k^9} = \frac{1}{k^2},$$

and since $\sum_{k=1}^{\infty} k^{-2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^{\infty} \frac{k^7}{k^9 + 3}$$

converges by the Direct Comparison Test.

5a Since $\ln k$ and k are monotone increasing functions for $k \geq 2$, it follows that

$$\frac{1}{k \ln^2 k}$$

is monotone decreasing (i.e. nonincreasing) for $k \geq 2$. Also

$$\lim_{k \to \infty} \frac{1}{k \ln^2 k} = 0,$$

and so by the Alternating Series Test the series converges.

5b Since

$$\lim_{k \to \infty} \left| (-1)^k \left(1 - \frac{2}{k}\right) \right| = \lim_{k \to \infty} \left(1 - \frac{2}{k}\right) = 1 \neq 0,$$

the series diverges by the Divergence Test.