# Math 141 Exam #2 Key (Summer 2013)

**1a** Let $u(x) = x$ and $v'(z) = \cos 8x$. Then $u'(x) = 1$ and $v(x) = \frac{1}{8}\sin 8x$, and so

$$\int x\cos 8x\,dx = \frac{x}{8}\sin 8x - \int \frac{1}{8}\sin 8x\,dx = \frac{x}{8}\sin 8x + \frac{1}{64}\cos 8x + c.$$

**1b** Let $u(x) = e^{-x}$ and $v'(x) = \sin(4x)$. Then $u'(x) = -e^{-x}$ and $v(x) = -\frac{1}{4}\cos(4x)$, and so

$$\int e^{-x}\sin(4x)\,dx = -\frac{1}{4}e^{-x}\cos(4x) - \frac{1}{4}\int e^{-x}\cos(4x)\,dx + c.$$

For the integral on the right, let $u(x) = e^{-x}$ and $v'(x) = \cos(4x)$. Then $u'(x) = -e^{-x}$ and $v(x) = \frac{1}{4}\sin(4x)$, so that

$$\int e^{-x}\sin(4x)\,dx = -\frac{1}{4}e^{-x}\cos(4x) - \frac{1}{4}\left[\frac{1}{4}e^{-x}\sin(4x) + \frac{1}{4}\int e^{-x}\sin(4x)\,dx\right] + c,$$

and hence

$$\int e^{-x}\sin(4x)\,dx = -\frac{4}{17}e^{-x}\cos(4x) - \frac{1}{17}e^{-x}\sin(4x) + c.$$

**1c** For $z \in \left(\frac{2}{\sqrt{3}}, 2\right)$ we have

$$(\sec^{-1} z)' = \frac{1}{z\sqrt{z^2-1}}.$$

Let $u(z) = \sec^{-1} z$ and $v'(z) = z$. Then $u'(z) = 1/\left(z\sqrt{z^2-1}\right)$ and $v(z) = \frac{1}{2}z^2$, and so

$$\int_{2/\sqrt{3}}^{2} \sec^{-1}(z)dz = \left[\tfrac{1}{2}z^2\sec^{-1}(z)\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2} \frac{z}{2\sqrt{z^2-1}}dz.$$

Let $w = z^2 - 1$ in the integral at right. We obtain

$$\int_{2/\sqrt{3}}^{2} \sec^{-1}(z)dz = 2\sec^{-1}(2) - \frac{2}{3}\sec^{-1}\left(2/\sqrt{3}\right) - \frac{1}{4}\int_{1/3}^{3} w^{-1/2}dw$$

$$= \frac{2\pi}{3} - \frac{\pi}{9} - \frac{1}{4}\left(\sqrt{3} - \sqrt{1/3}\right) = \frac{5\pi}{9} - \frac{1}{\sqrt{3}}.$$

**2a** We have

$$\int \sin^7(x)\cos^3(x)\,dx = \int [1-\cos^2(x)]^3\cos^3(x)\sin(x)\,dx,$$

and so if we let $u = \cos(x)$, so that $\sin(x)\,dx$ is replaced by $-du$ by the Substitution Rule, we obtain

$$\int \sin^7(x)\cos^3(x)\,dx = -\int (1-u^2)^3u^3\,du = \int (u^3 - 3u^5 + 3u^7 - u^9)\,du$$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^6 + \frac{3}{8}u^8 - \frac{1}{10}u^{10} + c$$

$$= \frac{1}{4}\cos^4 - \frac{1}{2}\cos^6 + \frac{3}{8}\cos^8 - \frac{1}{10}\cos^{10} + c.$$

Alternatively: write

$$\int \sin^7(x)[1-\sin^2(x)]\cos(x)\,dx,$$

and let $u = \sin(x)$ to obtain

$$\int u^7(1-u^2)\,du = \frac{1}{8}u^8 - \frac{1}{10}u^{10} + c = \frac{1}{8}\sin^8(x) - \frac{1}{10}\sin^{10}(x) + c.$$

**2b** We have

$$\int \frac{\csc^4 t}{\cot^2 t}dt = \int \csc^2 t \cdot \frac{\cot^2 t + 1}{\cot^2 t}dt.$$

Substitute $u = \cot t$, so that formally $\csc^2 t\,dt = -du$, and we obtain

$$\int \csc^2 t \cdot \frac{\cot^2 t+1}{\cot^2 t}dt = -\int \frac{u^2+1}{u^2}du = -\int (1+u^{-2})du = -u + \frac{1}{u} + c$$
$$= -\cot t + \tan t + c.$$

**2c** We have

$$\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2(\varphi)-1}\,d\varphi = \int_{-\pi/3}^{\pi/3} \sqrt{\tan^2(\varphi)}\,d\varphi = \int_{-\pi/3}^{\pi/3} |\tan(\varphi)|\,d\varphi$$
$$= \int_{-\pi/3}^{0} [-\tan(\varphi)]\,d\varphi + \int_0^{\pi/3} \tan(\varphi)\,d\varphi$$
$$= -\big[\ln|\sec(\varphi)|\big]_{-\pi/3}^{0} + \big[\ln|\sec(\varphi)|\big]_0^{\pi/3}$$
$$= 2\ln(2) = \ln(4).$$

**3a** Let $x - \frac{1}{3}\tan\theta$. Formally we obtain $dx = \frac{1}{3}\sec^2\theta d\theta$, and also $\sqrt{9x^2+1} = \sec\theta$. Running through the usual trigonometric substitution process yields

$$\int_0^{1/3} \frac{1}{(9x^2+1)^{3/2}}dx = \frac{\sqrt{2}}{6}.$$

**3b** Let $x = 11\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that $dx$ is replaced with $11\cos\theta\,d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

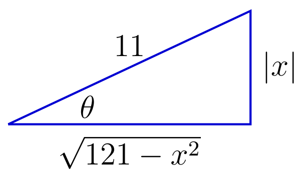
Now,

$$\int \sqrt{121 - x^2}\,dx = \int \sqrt{121 - 121\sin^2\theta} \cdot 11\cos\theta\,d\theta = \int 121\cos\theta\sqrt{1 - \sin^2\theta}\,d\theta$$
$$= 121\int \cos\theta\sqrt{\cos^2\theta}\,d\theta = 121\int \cos^2\theta\,d\theta,$$

and with the deft use of the given formula for $\int \cos^n\theta\,d\theta$ we obtain

$$\int \sqrt{121 - x^2}\,dx = 121\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int (1)\,d\theta\right) = \frac{121}{2}\cos\theta\sin\theta + \frac{121}{2}\theta + c.$$

From $x = 11\sin\theta$ comes $\sin\theta = x/11$, so $\theta = \sin^{-1}(x/11)$ and $\theta$ may be characterized as an angle in the right triangle




Note that $x \geq 0$ if $\theta \in [0, \pi/2]$, and $x < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{121 - x^2}/11$, and therefore

$$\int \sqrt{121 - x^2}\,dx = \frac{121}{2} \cdot \frac{\sqrt{121 - x^2}}{11} \cdot \frac{x}{11} + \frac{121}{2}\sin^{-1}\left(\frac{x}{11}\right) + c$$
$$= \frac{x\sqrt{121 - x^2}}{2} + \frac{121}{2}\sin^{-1}\left(\frac{x}{11}\right) + c.$$

**4a** We have

$$\frac{y+1}{y^3 + 3y^2 - 18y} = \frac{y+1}{y(y+6)(y-3)} = \frac{A}{y} + \frac{B}{y+6} + \frac{C}{y-3},$$

whence we obtain

$$y + 1 = (A + B + C)y^2 + (3A - 3B + 6C)y - 18A,$$

which results in the system

$$\begin{cases} A + B + C = 0 \\ 3A - 3B + 6C = 1 \\ -18A = 1 \end{cases}$$

Solving the system yields $(A, B, C) = \left(-\frac{1}{18}, -\frac{5}{54}, \frac{4}{27}\right)$. Hence

$$\int \frac{y+1}{y^3 + 3y^2 - 18y}dy = \int \left(-\frac{1}{18y} - \frac{5}{54(y+6)} + \frac{4}{27(y-3)}\right)dy$$
$$= -\frac{1}{18}\ln|y| - \frac{5}{24}\ln|y+6| + \frac{4}{27}\ln|y-3| + c.$$

**4b** Again start with a decomposition, noting that $x^2+2x+6$ is an irreducible quadratic:

$$\begin{aligned}\int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15-2/5}{x^2+2x+6}\right) dx \\ &= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{(x+1)^2+5}\,dx. \qquad (1)\end{aligned}$$

For the remaining integral, let $u = x+1$ to obtain

$$\int \frac{x+6}{(x+1)^2+5}\,dx = \int \frac{u+5}{u^2+5}\,du = \int \frac{u}{u^2+5}\,du + 5\int \frac{1}{u^2+(\sqrt{5})^2}\,du \qquad (2)$$

Letting $w = u^2+5$ in the first integral in (2), and using Formula (12) for the second, we next get

$$\begin{aligned}\int \frac{x+6}{(x+1)^2+5}\,dx &= \int \frac{1/2}{w}\,dw + 5\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln|w| + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2}\ln(u^2+5) + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln[(x+1)^2+5] + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\end{aligned}$$

Returning to (1),

$$\begin{aligned}\int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right] \\ &= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c.\end{aligned}$$

**5a** Letting $u = 2x-3$, we have

$$\begin{aligned}\int_{-\infty}^{1} \frac{1}{(2x-3)^2}\,dx &= \lim_{a\to-\infty}\int_a^1 \frac{1}{(2x-3)^2}\,dx = \lim_{a\to-\infty}\int_{2a-3}^{-1} \frac{1/2}{u^2}\,du \\ &= \lim_{a\to-\infty}\frac{1}{2}\left[-\frac{1}{u}\right]_{2a-3}^{-1} = \lim_{a\to-\infty}\frac{1}{2}\left(1+\frac{1}{2a-3}\right) = \frac{1}{2}.\end{aligned}$$

**5b** We must evaluate $\int_0^1 1/(x-1)\,dx$ and $\int_1^4 1/(x-1)\,dx$, if possible. By definition,

$$\begin{aligned}\int_0^1 \frac{1}{x-1}\,dx &= \lim_{b\to1^-}\int_0^b \frac{1}{x-1}\,dx = \lim_{b\to1^-}[\ln|x-1|]_0^b = \lim_{b\to1^-}(\ln|b-1| - \ln|-1|) \\ &= \lim_{b\to1^-}\ln(1-b) = -\infty.\end{aligned}$$

So $\int_0^1 1/(x-1)\,dx$ diverges, and therefore $\int_0^4 1/(x-1)\,dx$ also diverges.

**6** The volume of the solid is

$$\mathcal{V} = \int_2^\infty \pi(x^{-2})^2 dx = \pi \lim_{b\to\infty} \int_2^b \frac{1}{x^4} dx = \pi \lim_{b\to\infty} \left[-\tfrac{1}{3}x^{-3}\right]_2^b = \pi \lim_{b\to\infty} \tfrac{1}{3}(2^{-3} - b^{-3})$$

$$= \pi \cdot \tfrac{1}{3}(2^{-3} - 0) = \frac{\pi}{24}.$$