Math 141 Exam #1 Key (Summer 2013)

1. We have

$$f(2) = 2(2)^3 + 2 - 12 = 18 - 12 = 6,$$

and from $f'(x) = 6x^2 + 1$ we find that $f'(2) = 25 \neq 0$. Now, clearly f is differentiable on $(-\infty, \infty)$, and since $f' > 0$ on $(-\infty, \infty)$ we conclude that f is everywhere increasing and therefore one-to-one. By the appropriate theorem we then obtain

$$(f^{-1})'(6) = \frac{1}{f'(2)} = \frac{1}{25}.$$

2. Let f_1 be the restriction of f to the interval $[4, \infty)$. That is, $f_1(x) = f(x)$ for $x \geq 4$. Then f_1 is a one-to-one function and thus has an inverse f_1^{-1}. To find f_1^{-1} set $y = f_1(x)$, so that $y = (x-4)^2$ for $x \geq 4$. Then

$$\sqrt{y} = |x - 4| = x - 4,$$

whence $x = 4 + \sqrt{y}$. Since $y = f_1(x)$ if and only if $x = f_1^{-1}(y)$, we obtain $f_1^{-1}(y) = 4 + \sqrt{y}$.

Next, let f_2 be the restriction of f to the interval $(-\infty, 4]$. That is, $f_2(x) = f(x)$ for $x \leq 4$. Then f_2 is a one-to-one function and has an inverse f_2^{-1}. To find f_2^{-1} set $y = f_2(x)$, so that $y = (x-4)^2$ for $x \leq 4$. Then

$$\sqrt{y} = |x - 4| = -(x - 4) = 4 - x,$$

whence $x = 4 - \sqrt{y}$. Since $y = f_2(x)$ if and only if $x = f_2^{-1}(y)$, we obtain $f_2^{-1}(y) = 4 - \sqrt{y}$.

We have now found that there are two (local) inverses associated with f: the function f_1^{-1} given by

$$f_1^{-1}(y) = 4 + \sqrt{y}$$

with $\mathrm{Dom}(f_1^{-1}) = \mathrm{Ran}(f_1) = [0, \infty)$, and f_2^{-1} given by

$$f_2^{-1}(y) = 4 - \sqrt{y}$$

with $\mathrm{Dom}(f_2^{-1}) = \mathrm{Ran}(f_2) = [0, \infty)$.

3a. $f'(x) = \dfrac{2e^{2x}}{e^{2x} + 3}$

3b. $\mathrm{Dom}(g) = (0, \infty)$, and for all $x > 0$ we have

$$g(x) = x^{\ln(x^3)} = \exp\Big(\ln\Big(x^{\ln(x^3)}\Big)\Big) = \exp\big(\ln(x^3)\ln(x)\big) = \exp\big(3\ln^2(x)\big),$$

and thus

$$g'(x) = \exp\big(3\ln^2(x)\big) \cdot \big(3\ln^2(x)\big)' = x^{\ln(x^3)} \cdot \frac{6\ln(x)}{x} = \frac{6x^{\ln(x^3)}\ln(x)}{x}.$$

3c. For x such that $\tan(x) > 0$ we have

$$h(x) = (\tan x)^{\cos x} = \exp(\ln((\tan x)^{\cos x})) = \exp(\cos x \cdot \ln(\tan x)),$$

and thus

$$\begin{aligned} h'(x) &= \exp(\cos x \cdot \ln(\tan x)) \cdot (\cos x \cdot \ln(\tan x))' \\ &= \exp(\cos x \cdot \ln(\tan x)) \cdot \left(\cos x \cdot \frac{\sec^2 x}{\tan x} - \sin x \cdot \ln(\tan x) \right) \\ &= (\tan x)^{\cos x} \left(\csc x - \ln(\tan x)^{\sin x} \right) \end{aligned}$$

3d. $k'(x) = \dfrac{7}{(4 - x^5)\ln(3)} \cdot (4 - x^5)' = -\dfrac{35x^4}{(4 - x^5)\ln(3)}$

3e. $\ell'(x) = \dfrac{1}{\sqrt{1 - (e^{-2x})^2}} \cdot (e^{-2x})' = -\dfrac{2e^{-2x}}{\sqrt{1 - e^{-4x}}}$

3f. $p'(x) = -\dfrac{1}{1 + (\sqrt{x}\,)^2} \cdot (\sqrt{x}\,)' = -\dfrac{1}{1 + x} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x}(1 + x)}$

4a. $\displaystyle\int (3e^{-8x} - 8e^{11x})dx = -\frac{3}{8}e^{-8x} - \frac{8}{11}e^{11x} + c$

4b. $\displaystyle\int \frac{4}{3 - 10x}dx = -\frac{2}{5}\ln|3 - 10x| + c$

4c. Let $u = x^4$, so by the Substitution Rule we replace $x^3\,dx$ with $\frac{1}{4}du$ to get

$$\int x^3 9^{x^4}\,dx = \int \frac{1}{4}9^u\,du = \frac{1}{4} \cdot \frac{9^u}{\ln(9)} + c = \frac{9^{x^4}}{4\ln(9)} + c.$$

5a. Let $u = \ln(x)$, so when $x = 1$ we have $u = \ln(1) = 0$, and when $x = 3e$ we have $u = \ln(3e)$. Now, by the Substitution Rule we replace $\frac{1}{x}dx$ with du to get

$$\int_0^{\ln(3e)} \frac{e^u}{2}\,du = \left[\frac{1}{2}e^u\right]_0^{\ln(3e)} = \frac{1}{2}(e^{\ln(3e)} - e^0) = \frac{3e - 1}{2}.$$

5b. We have

$$5\int_2^{2\sqrt{3}} \frac{1}{z^2 + 2^2}dz = 5\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_2^{2\sqrt{3}} = \frac{5}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right] = \frac{5}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{5\pi}{24}.$$

6. For all $x > 0$ we have

$$\left(\frac{2}{3x}\right)^{8/x} = \exp\left[\ln\left(\frac{2}{3x}\right)^{8/x}\right] = \exp\left[\frac{8}{x}\ln\left(\frac{2}{3x}\right)\right] = \exp\left(\frac{8\ln(2/3x)}{x}\right).$$

The functions $f(x) = 8\ln(2/3x)$ and $g(x) = x$ are differentiable on $(0,\infty)$, and $g'(x) = 1 \neq 0$ for all $x \in (0,\infty)$. Since $g(x) \to \infty$ as $x \to \infty$, and

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{-8/x}{1} = 0,$$

by L'Hôpital's Rule we obtain

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{8\ln(2/3x)}{x} = 0$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to\infty}\left(\frac{2}{3x}\right)^{8/x} = \lim_{x\to\infty}\exp\left(\frac{8\ln(2/3x)}{x}\right) = \exp\left(\lim_{x\to\infty}\frac{8\ln(2/3x)}{x}\right) = \exp(0) = 1.$$

7. We find that $x^{20}, 1.001^x \to \infty$ as $x \to \infty$. To determine which grows faster, x^{20} or 1.001^x, we must evaluate

$$\lim_{x\to\infty}\frac{1.001^x}{x^{20}}.$$

If the limit equals 0 then x^{20} grows faster, and if the limit equals ∞, then 1.001^x grows faster. We have

$$\begin{aligned}\lim_{x\to\infty}\ln\left(\frac{1.001^x}{x^{20}}\right) &= \lim_{x\to\infty}\left(\ln 1.001^x - \ln x^{20}\right) = \lim_{x\to\infty}(x\ln 1.001 - 20\ln x)\\ &= \lim_{x\to\infty} x\left(\ln 1.001 - \frac{20\ln x}{x}\right),\end{aligned}$$

where we easily find by L'Hôpital's Rule that

$$\lim_{x\to\infty}\frac{20\ln x}{x} = \lim_{x\to\infty}\frac{20/x}{1} = 0,$$

so that

$$\lim_{x\to\infty}\left(\ln 1.001 - \frac{20\ln x}{x}\right) = \ln 1.001 > 0$$

and therefore

$$\lim_{x\to\infty}\ln\left(\frac{1.001^x}{x^{20}}\right) = \lim_{x\to\infty} x\left(\ln 1.001 - \frac{20\ln x}{x}\right) = (\infty)(\ln 1.001) = \infty.$$

Now,

$$\lim_{x\to\infty}\frac{1.001^x}{x^{20}} = \lim_{x\to\infty}\exp\left[\ln\left(\frac{1.001^x}{x^{20}}\right)\right] = \exp\left[\lim_{x\to\infty}\ln\left(\frac{1.001^x}{x^{20}}\right)\right] = \exp(\infty) = \infty,$$

so 1.001^x grows faster than x^{20} and we write $1.001^x \gg x^{20}$.