Math 141 Exam #3 Key (Summer 2012)

1a. Since $\ln k$ and k are monotone increasing functions for $k \geq 2$, it follows that

$$\frac{1}{k \ln^2 k}$$

is monotone decreasing (i.e. nonincreasing) for $k \geq 2$. Also

$$\lim_{k\to\infty} \frac{1}{k \ln^2 k} = 0,$$

and so by the Alternating Series Test the series converges.

1b. Since

$$\lim_{k\to\infty} \left| (-1)^k \left(1 - \frac{2}{k}\right) \right| = \lim_{k\to\infty} \left(1 - \frac{2}{k}\right) = 1 \neq 0,$$

the series diverges by the Divergence Test.

2a. Applying the Ratio Test,

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(x+1)^{k+1}}{8^{k+1}} \cdot \frac{8^k}{(x+1)^k} \right| = \lim_{k\to\infty} \frac{|x+1|}{8} = \frac{|x+1|}{8},$$

so the series converges if $|x+1|/8 < 1$, implying $-8 < x+1 < 8$ and thus $-9 < x < 7$. It remains to test the endpoints.

When $x = 7$ the series becomes,

$$\lim_{k\to\infty} \left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty} \left(\frac{7+1}{8}\right)^k = \lim_{k\to\infty} (1) = 1 \neq 0,$$

so the series diverges by the Divergence Test.

When $x = -9$ the series becomes,

$$\lim_{k\to\infty} \left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty} \left(\frac{-9+1}{8}\right)^k = \lim_{k\to\infty} (-1)^k \neq 0,$$

so again the series diverges. Therefore the interval of convergence is $(-9, 7)$, and the radius of convergence is $|-9-7|/2 = 8$.

2b. Applying the Ratio Test,

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(2x+3)^{k+1}}{6(k+1)} \cdot \frac{6k}{(2x+3)^k} \right| = \lim_{k\to\infty} \frac{k|2x+3|}{k+1} = |2x+3|,$$

so the series converges if $-1 < 2x+3 < 1$, implying $-2 < x < -1$.

When $x = -2$ the series becomes

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{6k},$$

which converges by the Alternating Series Test. When $x = -1$ the series becomes

$$\sum_{k=1}^{\infty} \frac{1}{6k},$$

which diverges since

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

diverges. Interval of convergence is $[-2, -1)$, radius of convergence is $\frac{1}{2}$.

3. We manipulate to obtain

$$g(x) = 5 \cdot \frac{1}{1-6x} = 5 \sum_{k=0}^{\infty} (6x)^k = \sum_{k=0}^{\infty} 5(6x)^k,$$

which converges if and only if $|6x| < 1$, so the interval of convergence is $\left(-\frac{1}{6}, \frac{1}{6}\right)$.

4. Use the geometric series given in the previous problem to get

$$f(x) = \frac{1}{1 - (\sqrt{x} - 7)} = \frac{1}{8 - \sqrt{x}}.$$

The series converges if and only if $|\sqrt{x} - 7| < 1$, which solves to give $6 < \sqrt{x} < 8$ and then $36 < x < 64$. So interval of convergence is $(36, 64)$.

5. We evaluate $b_k = (2k+1)^{-3}$ for successive values of k until we obtain a number less than 10^{-4}:

$$\begin{aligned} b_0 &= 1 & b_6 &= 13^{-3} \approx 4.55 \times 10^{-4} \\ b_1 &= 1/27 & b_7 &= 15^{-3} \approx 2.96 \times 10^{-4} \\ b_2 &= 1/125 & b_8 &= 17^{-3} \approx 2.04 \times 10^{-4} \\ b_3 &= 7^{-3} \approx 2.92 \times 10^{-3} & b_9 &= 19^{-3} \approx 1.46 \times 10^{-4} \\ b_4 &= 9^{-3} \approx 1.37 \times 10^{-3} & b_{10} &= 21^{-3} \approx 1.08 \times 10^{-4} \\ b_5 &= 11^{-3} \approx 7.51 \times 10^{-4} & b_{11} &= 23^{-3} \approx 8.22 \times 10^{-5} \end{aligned}$$

Thus, by the Remainder Theorem we have

$$R_{10} = |s - s_{10}| \le b_{11} \approx 8.22 \times 10^{-5} < 10^{-4},$$

which is to say that the approximation

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \approx \sum_{k=0}^{10} \frac{(-1)^k}{(2k+1)^3} = s_{10} = 1^{-3} - 3^{-3} + \cdots + 21^{-3} \approx 1.0277$$

has an absolute error that is less than 10^{-4}.

6a. We have

$$\frac{4^0}{0!}x^0 - \frac{4^2}{2!}x^2 + \frac{4^4}{4!}x^4 - \frac{4^6}{6!}x^6 + \cdots = 1 - \frac{16}{2}x^2 + \frac{256}{24}x^4 - \frac{4096}{720}x^6 + \cdots$$
$$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \cdots$$

6b. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k 4^{2k}}{(2k)!} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k (4x)^{2k}}{(2k)!}.$

6c. Use the Ratio Test to find that the interval of convergence is $(-\infty, \infty)$.

7. Since the limit takes x toward 0 we use the Maclaurin series for $\cos x$ and e^x:

$$\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{1 - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}}{1 + x - \sum_{k=0}^{\infty} \frac{x^k}{k!}} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)}{1 + x - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \cdots}{-\frac{x^2}{2} - \frac{x^3}{6} - \cdots} = \lim_{x\to 0}\left(-1 + \frac{x^3}{6} - \frac{x^4}{24} + \cdots\right) = -1.$$

Note that long division is employed to obtain the penultimate equality.

8. Because the Maclaurin series for $\sin(x)$ is everywhere convergent it can be multiplied by $1/x$ termwise:

$$\frac{\sin(x)}{x} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}.$$

Now, the new series we obtain is also everywhere convergent, so it can be integrated termwise:

$$\int_0^{0.15} \frac{\sin(x)}{x}\,dx = \int_0^{0.15} \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}\right] dx = \sum_{k=0}^{\infty} \left[\int_0^{0.15} \frac{(-1)^k x^{2k}}{(2k+1)!}\,dx\right]$$
$$= \sum_{k=0}^{\infty} \left[\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}\right]_0^{0.15} = \sum_{k=0}^{\infty} \frac{(-1)^k (0.15)^{2k+1}}{(2k+1)(2k+1)!}$$
$$= \frac{0.15}{(1)(1)} - \frac{0.15^3}{(3)(3!)} + \frac{0.15^5}{(5)(5!)} - \frac{0.15^7}{(7)(7!)} + \cdots$$
$$\approx 0.15 - 1.875 \times 10^{-4} + 1.266 \times 10^{-7} - \cdots.$$

The Remainder Theorem assures us that if we estimate the value of $\int_0^{0.15} \sin(x)/x\,dx$ by

$$0.15 - 1.875 \times 10^{-4} \approx 0.1498$$

then the error will be no greater than 1.266×10^{-7}, and this is certainly within our accepted tolerance of 10^{-4}! Therefore our estimate is 0.1498.

9. From $x = \sqrt{t} + 4$ comes $\sqrt{t} = x - 4$. Putting this into $y = 3\sqrt{t}$ gives $y = 3(x - 4)$. Note that this will not be a line, since $0 \le t \le 16$ implies $0 \le \sqrt{t} \le 4$, and this means $0 \le x - 4 \le 4$. That is, we have

$$y = 3x - 12, \quad 4 \le x \le 8,$$

which is a line segment.

10. $(2, 2\pi/3)$, $(2, -4\pi/3)$, $(-2, -\pi/3)$, among other possibilities.

11. We have $r = f(\theta)$ with $f(\theta) = 8\sin\theta$. The slope m of the curve at $(4, 5\pi/6)$ is

$$\begin{aligned} m &= \frac{f'(5\pi/6)\sin(5\pi/6) + f(5\pi/6)\cos(5\pi/6)}{f'(5\pi/6)\cos(5\pi/6) - f(5\pi/6)\sin(5\pi/6)} \\ &= \frac{8\cos(5\pi/6)\sin(5\pi/6) + 8\sin(5\pi/6)\cos(5\pi/6)}{8\cos(5\pi/6)\cos(5\pi/6) - 8\sin(5\pi/6)\sin(5\pi/6)} \\ &= \frac{2\cos(5\pi/6)\sin(5\pi/6)}{\cos^2(5\pi/6) - \sin^2(5\pi/6)} = \frac{2(-\sqrt{3}/2)(1/2)}{(-\sqrt{3}/2)^2 - (1/2)^2} = -\sqrt{3} \end{aligned}$$

12. Here $r = f(\theta)$ with $f(\theta) = 3 + 5\sin\theta$. The curve is generated for $\theta \in [0, 2\pi)$, so we find all $0 \le \theta < 2\pi$ for which

$$f'(\theta)\sin\theta + f(\theta)\cos\theta = 0,$$

which gives

$$5\cos\theta\sin\theta + \cos\theta(3 + 5\sin\theta) = 0.$$

Factoring results in the equation

$$(10\sin\theta + 3)\cos\theta = 0,$$

so either $10\sin\theta + 3 = 0$ or $\cos\theta = 0$. The latter equation has solutions $\theta = \pi/2, 3\pi/2$. The former equation gives $\sin\theta = -3/10$, and though the angle $\sin^{-1}(-3/10)$ is not in the interval $[0, 2\pi)$, the angles

$$2\pi + \sin^{-1}(-3/10) \quad \text{and} \quad \pi - \sin^{-1}(-3/10)$$

both are. You *must* be up on your basic trigonometry to figure this out!

Therefore the points on the curve $r = 3 + 5\sin\theta$ where the tangent line is horizontal are:

$$\left(8, \tfrac{\pi}{2}\right), \quad \left(-2, \tfrac{3\pi}{2}\right), \quad \left(\tfrac{3}{2}, 2\pi + \sin^{-1}(-\tfrac{3}{10})\right), \quad \left(\tfrac{3}{2}, \pi - \sin^{-1}(-\tfrac{3}{10})\right).$$