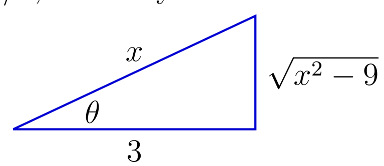
Math 141 Exam #2 Key (Summer 2012)

1a. Let $x = 3\sec\theta$, so that dx is replaced with $3\sec\theta\tan\theta\,d\theta$ as part of the substitution. Since $x > 3$ we have $\sec\theta > 1$, which implies $\theta \in (0, \pi/2)$ and so $\tan\theta > 0$. Now, making use of the given formula for $\int \tan^n\theta\,d\theta$, we obtain

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,d\theta = \int 3\sqrt{\tan^2\theta}\cdot\tan\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta = 3\left(\tan\theta - \int 1\,d\theta\right) = 3\tan\theta - 3\theta + c.$$

From $x = 3\sec\theta$ comes $\sec\theta = x/3$, so θ may be characterized as an angle in the right triangle



From this triangle we see that

$$\tan\theta = \frac{\sqrt{x^2-9}}{3}, \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right)$$

and therefore

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\tan^{-1}\left(\frac{\sqrt{x^2-9}}{3}\right) + c.$$

1b. Let $x = 11\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that dx is replaced with $11\cos\theta\,d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

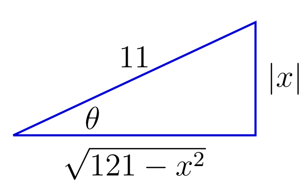
Now,

$$\int \sqrt{121 - x^2}\,dx = \int \sqrt{121 - 121\sin^2\theta}\cdot 11\cos\theta\,d\theta = \int 121\cos\theta\sqrt{1-\sin^2\theta}\,d\theta$$
$$= 121\int \cos\theta\sqrt{\cos^2\theta}\,d\theta = 121\int \cos^2\theta\,d\theta,$$

and with the deft use of the given formula for $\int \cos^n\theta\,d\theta$ we obtain

$$\int \sqrt{121-x^2}\,dx = 121\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int (1)\,d\theta\right) = \frac{121}{2}\cos\theta\sin\theta + \frac{121}{2}\theta + c.$$

From $x = 11\sin\theta$ comes $\sin\theta = x/11$, so $\theta = \sin^{-1}(x/11)$ and θ may be characterized as an angle in the right triangle



Note that $x \geq 0$ if $\theta \in [0, \pi/2]$, and $x < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{121 - x^2}/11$, and therefore

$$\begin{aligned} \int \sqrt{121 - x^2}\, dx &= \frac{121}{2} \cdot \frac{\sqrt{121 - x^2}}{11} \cdot \frac{x}{11} + \frac{121}{2} \sin^{-1}\!\left(\frac{x}{11}\right) + c \\ &= \frac{x\sqrt{121 - x^2}}{2} + \frac{121}{2} \sin^{-1}\!\left(\frac{x}{11}\right) + c. \end{aligned}$$

2a. This is a job for partial fraction decomposition:

$$\begin{aligned} \int \frac{3}{x^3 - x^2 - 12x}\, dx &= \int \frac{3}{x(x-4)(x+3)}\, dx = \int \left(\frac{-1/4}{x} + \frac{3/28}{x-4} + \frac{1/7}{x+3} \right) dx \\ &= -\frac{1}{4} \ln|x| + \frac{3}{28} \ln|x - 4| + \frac{1}{7} \ln|x + 3| + c. \end{aligned}$$

2b. Again start with a decomposition, noting that $x^2 + 2x + 6$ is an irreducible (i.e. unfactorable) quadratic:

$$\begin{aligned} \int \frac{2}{(x-4)(x^2 + 2x + 6)}\, dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2 + 2x + 6} \right) dx \\ &= \frac{1}{15} \ln|x - 4| - \frac{1}{15} \int \frac{x + 6}{(x+1)^2 + 5}\, dx. \end{aligned} \tag{1}$$

For the remaining integral, let $u = x + 1$ to obtain

$$\int \frac{x+6}{(x+1)^2 + 5}\, dx = \int \frac{u + 5}{u^2 + 5}\, du = \int \frac{u}{u^2 + 5}\, du + 5 \int \frac{1}{u^2 + (\sqrt{5})^2}\, du \tag{2}$$

Letting $w = u^2 + 5$ in the first integral in (2), and using Formula (13) for the second, we next get

$$\begin{aligned} \int \frac{x+6}{(x+1)^2 + 5}\, dx &= \int \frac{1/2}{w}\, dw + 5 \cdot \frac{1}{\sqrt{5}} \tan^{-1}\!\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2} \ln|w| + \sqrt{5} \tan^{-1}\!\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2} \ln(u^2 + 5) + \sqrt{5} \tan^{-1}\!\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2} \ln[(x+1)^2 + 5] + \sqrt{5} \tan^{-1}\!\left(\frac{x+1}{\sqrt{5}}\right) + c \end{aligned}$$

Returning to (1),

$$\begin{aligned} \int \frac{2}{(x-4)(x^2 + 2x + 6)}\, dx &= \frac{\ln|x-4|}{15} - \frac{1}{15} \left[\frac{\ln[(x+1)^2 + 5]}{2} + \sqrt{5} \tan^{-1}\!\left(\frac{x+1}{\sqrt{5}}\right) + c \right] \\ &= \frac{\ln|x-4|}{15} - \frac{\ln(x^2 + 2x + 6)}{30} - \frac{\sqrt{5}}{15} \tan^{-1}\!\left(\frac{x+1}{\sqrt{5}}\right) + c. \end{aligned}$$

3a. Letting $u = 2x - 3$, we have

$$\int_{-\infty}^{1} \frac{1}{(2x-3)^2}\,dx = \lim_{a\to-\infty} \int_{a}^{1} \frac{1}{(2x-3)^2}\,dx = \lim_{a\to-\infty} \int_{2a-3}^{-1} \frac{1/2}{u^2}\,du$$
$$= \lim_{a\to-\infty} \frac{1}{2}\left[-\frac{1}{u}\right]_{2a-3}^{-1} = \lim_{a\to-\infty} \frac{1}{2}\left(1 + \frac{1}{2a-3}\right) = \frac{1}{2}.$$

3b. We must evaluate $\int_0^1 1/(x-1)\,dx$ and $\int_1^4 1/(x-1)\,dx$, if possible. By definition,

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{b\to1^-} \int_0^b \frac{1}{x-1}\,dx = \lim_{b\to1^-} [\ln|x-1|]_0^b = \lim_{b\to1^-} (\ln|b-1| - \ln|-1|)$$
$$= \lim_{b\to1^-} \ln(1-b) = -\infty.$$

So $\int_0^1 1/(x-1)\,dx$ diverges, and therefore $\int_0^4 1/(x-1)\,dx$ also diverges.

4a. We have

$$\lim_{n\to\infty} \frac{5n^8}{\sqrt{36n^{16} - 10n^{10}}} = \lim_{n\to\infty} \frac{5n^8}{n^8\sqrt{36 - 10/n^6}} = \lim_{n\to\infty} \frac{5}{\sqrt{36 - 10/n^6}} = \frac{5}{\sqrt{36}} = \frac{5}{6}.$$

4b. First we evaluate

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp(\ln n^{1/n}) = \exp\left(\lim_{n\to\infty} \ln n^{1/n}\right)$$
$$= \exp\left(\lim_{n\to\infty} \frac{\ln n}{n}\right) \overset{LR}{=} \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) = \exp(0) = 1.$$

Now, consider the subsequence of $\{a_n\}_{n=1}^{\infty}$ that consists of the even-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k$ for $k \geq 1$. Then, using the fact that $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{n_k} n_k^{1/n_k} = \lim_{k\to\infty} (-1)^{2k}(2k)^{1/(2k)} = \lim_{k\to\infty} (2k)^{1/(2k)} = 1.$$

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k-1$ for $k \geq 1$. Then we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{2k-1}(2k-1)^{1/(2k-1)} = \lim_{k\to\infty} \left[-(2k-1)^{1/(2k-1)}\right] = -1.$$

Since $\{a_n\}$ has two subsequences with different limits, the sequence $\{a_n\}$ itself cannot converge. That is, $\{a_n\}$ diverges.

5. Starting by reindexing, we have

$$\sum_{k=2}^{\infty} \frac{3}{(-2)^k} = \sum_{k=0}^{\infty} \frac{3}{(-2)^{k+2}} = \sum_{k=0}^{\infty} \frac{3}{4}\left(-\frac{1}{2}\right)^k = \frac{3/4}{1-(-1/2)} = \frac{1}{2}.$$

6. Partial fraction decomposition gives

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2},$$

so series becomes

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

Now,

$$\begin{aligned} s_n &= \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\ &= \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\ &= \frac{1}{2} - \frac{1}{n+2}, \end{aligned}$$

so

$$\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2}.$$

7a. Since

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+25}} = 1 \neq 0,$$

the series diverges by the Divergence Test.

7b. Letting $u = -2x^2$, we have

$$\begin{aligned} \int_1^{\infty} xe^{-2x^2}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-2x^2}\,dx = \lim_{b\to\infty}\int_{-2}^{-2b^2} -\frac{1}{4}e^u\,du = \lim_{b\to\infty} -\frac{1}{4}[e^u]_{-2}^{-2b^2} \\ &= \lim_{b\to\infty} -\frac{1}{4}\left(e^{-2b^2} - e^{-2}\right) = -\frac{1}{4}(0 - e^{-2}) = \frac{e^{-2}}{4}, \end{aligned}$$

so the integral

$$\int_1^{\infty} xe^{-2x^2}\,dx$$

converges, and therefore the series converges by the Integral Test.

7c. Since

$$\lim_{k\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty}\left|\frac{[(k+1)!]^2}{[2(k+1)]!}\cdot\frac{(2k)!}{(k!)^2}\right| = \lim_{k\to\infty}\frac{(k+1)(k+1)}{(2k+1)(2k+2)} = \lim_{k\to\infty}\frac{k+1}{4k+2} = \frac{1}{4} < 1,$$

the series converges by the Ratio Test.

7d. Since

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{\frac{k^2}{2^k}} = \lim_{k\to\infty}\frac{k^{2/k}}{2} = \frac{1}{2} < 1,$$

the series converges by the Root Test.

7e. For each $k \geq 1$ we have

$$0 \leq \frac{\sin^2 k}{k\sqrt{k}} \leq \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}},$$

and since $\sum_{k=1}^{\infty} k^{-3/2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^{\infty}\frac{\sin^2 k}{k\sqrt{k}}$$

converges by the Direct Comparison Test.

7f. For each $k \geq 1$ we have

$$0 \leq \frac{k^7}{k^9+3} \leq \frac{k^7}{k^9} = \frac{1}{k^2},$$

and since $\sum_{k=1}^{\infty} k^{-2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^{\infty}\frac{k^7}{k^9+3}$$

converges by the Direct Comparison Test.