Math 141 Exam #1 Key (Summer 2012)

1. We have

$$f(2) = 2(2)^3 + 2 - 12 = 18 - 12 = 6,$$

and from $f'(x) = 6x^2 + 1$ we find that $f'(2) = 25 \neq 0$. Now, clearly f is differentiable on $(-\infty, \infty)$, and since $f' > 0$ on $(-\infty, \infty)$ we conclude that f is everywhere increasing and therefore one-to-one. By the appropriate theorem we then obtain

$$(f^{-1})'(6) = \frac{1}{f'(2)} = \frac{1}{25}.$$

2. Let f_1 be the restriction of f to the interval $[4, \infty)$. That is, $f_1(x) = f(x)$ for $x \geq 4$. Then f_1 is a one-to-one function and thus has an inverse f_1^{-1}. To find f_1^{-1} set $y = f_1(x)$, so that $y = (x-4)^2$ for $x \geq 4$. Then

$$\sqrt{y} = |x - 4| = x - 4,$$

whence $x = 4 + \sqrt{y}$. Since $y = f_1(x)$ if and only if $x = f_1^{-1}(y)$, we obtain $f_1^{-1}(y) = 4 + \sqrt{y}$.

Next, let f_2 be the restriction of f to the interval $(-\infty, 4]$. That is, $f_2(x) = f(x)$ for $x \leq 4$. Then f_2 is a one-to-one function and has an inverse f_2^{-1}. To find f_2^{-1} set $y = f_2(x)$, so that $y = (x-4)^2$ for $x \leq 4$. Then

$$\sqrt{y} = |x - 4| = -(x - 4) = 4 - x,$$

whence $x = 4 - \sqrt{y}$. Since $y = f_2(x)$ if and only if $x = f_2^{-1}(y)$, we obtain $f_2^{-1}(y) = 4 - \sqrt{y}$.

We have now found that there are two (local) inverses associated with f: the function f_1^{-1} given by

$$f_1^{-1}(y) = 4 + \sqrt{y}$$

with $\text{Dom}(f_1^{-1}) = \text{Ran}(f_1) = [0, \infty)$, and f_2^{-1} given by

$$f_2^{-1}(y) = 4 - \sqrt{y}$$

with $\text{Dom}(f_2^{-1}) = \text{Ran}(f_2) = [0, \infty)$.

3a. $f'(x) = \dfrac{2e^{2x}}{e^{2x} + 3}$

3b. $\text{Dom}(g) = (0, \infty)$, and for all $x > 0$ we have

$$g(x) = x^{\ln(x^3)} = \exp\Big(\ln\Big(x^{\ln(x^3)}\Big)\Big) = \exp\big(\ln(x^3)\ln(x)\big) = \exp\big(3\ln^2(x)\big),$$

and thus

$$g'(x) = \exp\big(3\ln^2(x)\big)\cdot\big(3\ln^2(x)\big)' = x^{\ln(x^3)} \cdot \frac{6\ln(x)}{x} = \frac{6x^{\ln(x^3)}\ln(x)}{x}.$$

3c. For x such that $\tan(x) > 0$ we have

$$h(x) = (\tan x)^{\cos x} = \exp(\ln((\tan x)^{\cos x})) = \exp(\cos x \cdot \ln(\tan x)),$$

and thus

$$\begin{aligned} h'(x) &= \exp(\cos x \cdot \ln(\tan x)) \cdot (\cos x \cdot \ln(\tan x))' \\ &= \exp(\cos x \cdot \ln(\tan x)) \cdot \left(\cos x \cdot \frac{\sec^2 x}{\tan x} - \sin x \cdot \ln(\tan x) \right) \\ &= (\tan x)^{\cos x} \big(\csc x - \ln(\tan x)^{\sin x} \big) \end{aligned}$$

3d. $k'(x) = \dfrac{7}{(4 - x^5)\ln(3)} \cdot (4 - x^5)' = -\dfrac{35x^4}{(4 - x^5)\ln(3)}$

3e. $\ell'(x) = \dfrac{1}{\sqrt{1 - (e^{-2x})^2}} \cdot (e^{-2x})' = -\dfrac{2e^{-2x}}{\sqrt{1 - e^{-4x}}}$

3f. $p'(x) = -\dfrac{1}{1 + (\sqrt{x}\,)^2} \cdot (\sqrt{x}\,)' = -\dfrac{1}{1 + x} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x}(1 + x)}$

4a. $\displaystyle\int (3e^{-8x} - 8e^{11x})dx = -\frac{3}{8}e^{-8x} - \frac{8}{11}e^{11x} + c$

4b. $\displaystyle\int \frac{4}{3 - 10x}dx = -\frac{4}{10}\ln|3 - 10x| + c$

4c. Let $u = x^4$, so by the Substitution Rule we replace $x^3\,dx$ with $\frac{1}{4}du$ to get

$$\int x^3 9^{x^4}\,dx = \int \frac{1}{4}9^u\,du = \frac{1}{4} \cdot \frac{9^u}{\ln(9)} + c = \frac{9^{x^4}}{4\ln(9)} + c.$$

5a. Let $u = \ln(x)$, so when $x = 1$ we have $u = \ln(1) = 0$, and when $x = 3e$ we have $u = \ln(3e)$. Now, by the Substitution Rule we replace $\frac{1}{x}dx$ with du to get

$$\int_0^{\ln(3e)} \frac{e^u}{2}\,du = \left[\frac{1}{2}e^u\right]_0^{\ln(3e)} = \frac{1}{2}(e^{\ln(3e)} - e^0) = \frac{3e - 1}{2}.$$

5b. We have

$$5\int_2^{2\sqrt{3}} \frac{1}{z^2 + 2^2}dz = 5\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_2^{2\sqrt{3}} = \frac{5}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right] = \frac{5}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{5\pi}{24}.$$

6. For all $x > 0$ we have

$$\left(\frac{1}{5x}\right)^{2/x} = \exp\left[\ln\left(\frac{1}{5x}\right)^{2/x}\right] = \exp\left[\frac{2}{x}\ln\left(\frac{1}{5x}\right)\right] = \exp\left(\frac{-2\ln(5x)}{x}\right).$$

The functions $f(x) = -2\ln(5x)$ and $g(x) = x$ are differentiable on $(0, \infty)$, and $g'(x) = 1 \neq 0$ for all $x \in (0, \infty)$. Since $g(x) \to \infty$ as $x \to \infty$, and

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-2/x}{1} = 0,$$

by L'Hôpital's Rule we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{-2\ln(5x)}{x} = 0$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to\infty}\left(\frac{1}{5x}\right)^{2/x} = \lim_{x\to\infty}\exp\left(\frac{-2\ln(5x)}{x}\right) = \exp\left(\lim_{x\to\infty}\frac{-2\ln(5x)}{x}\right) = \exp(0) = 1.$$

7a. Let $u(\theta) = \theta$ and $v'(\theta) = \sec^2(\theta)$. Then $u'(\theta) = 1$ and $v(\theta) = \tan(\theta)$, and so

$$\int \theta \sec^2(\theta)\,d\theta = \theta\tan(\theta) - \int \tan(\theta)\,d\theta = \theta\tan(\theta) - \ln|\sec(\theta)| + c.$$

7b. Let $u(x) = e^{-x}$ and $v'(x) = \sin(3x)$. Then $u'(x) = -e^{-x}$ and $v(x) = -\frac{1}{3}\cos(3x)$, and so

$$\int e^{-x}\sin(3x)\,dx = -\frac{1}{3}e^{-x}\cos(3x) - \frac{1}{3}\int e^{-x}\cos(3x)\,dx + c.$$

For the integral on the right, let $u(x) = e^{-x}$ and $v'(x) = \cos(3x)$. Then $u'(x) = -e^{-x}$ and $v(x) = \frac{1}{3}\sin(3x)$, so that

$$\int e^{-x}\sin(3x)\,dx = -\frac{1}{3}e^{-x}\cos(3x) - \frac{1}{3}\left[\frac{1}{3}e^{-x}\sin(3x) + \frac{1}{3}\int e^{-x}\sin(3x)\,dx\right] + c,$$

and hence

$$\int e^{-x}\sin(3x)\,dx = -\frac{1}{10}e^{-x}[3\cos(3x) + \sin(3x)] + c.$$

7c. Let $u(y) = \sin^{-1}(y)$ and $v'(y) = 1$. Then $u'(y) = 1/\sqrt{1-y^2}$ and $v(y) = y$, and so

$$\int_{1/2}^{\sqrt{3}/2} \sin^{-1}(y)\,dy = \left[y\sin^{-1}(y)\right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{y}{\sqrt{1-y^2}}\,dy \tag{1}$$

Now, let $w = 1 - y^2$ in the integral on the right-hand side of (1), so that by the Substitution Rule $y\,dy$ is replaced with $-\frac{1}{2}dw$. When $y = 1/2$ we have $w = 3/4$, and when $y = \sqrt{3}/2$ we have $w = 1/4$, and so

$$\int_{1/2}^{\sqrt{3}/2} \frac{y}{\sqrt{1-y^2}}\,dy = -\frac{1}{2}\int_{3/4}^{1/4} \frac{1}{\sqrt{w}}\,dw = -\frac{1}{2}\left[2\sqrt{w}\right]_{3/4}^{1/4} = \left[\sqrt{w}\right]_{1/4}^{3/4} = \frac{\sqrt{3}-1}{2}.$$

Returning to (1), we obtain

$$\begin{aligned}\int_{1/2}^{\sqrt{3}/2} \sin^{-1}(y)\,dy &= \left[y\sin^{-1}(y)\right]_{1/2}^{\sqrt{3}/2} - \frac{\sqrt{3}-1}{2} \\ &= \frac{\sqrt{3}}{2}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \frac{\sqrt{3}-1}{2} \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\cdot\frac{\pi}{6} - \frac{\sqrt{3}-1}{2} = \left(\frac{2\sqrt{3}-1}{12}\right)\pi - \frac{\sqrt{3}-1}{2}\end{aligned}$$

8a. We have

$$\int \sin^7(x)\cos^3(x)\,dx = \int (1-\cos^2(x))^3\cos^3(x)\sin(x)\,dx,$$

and so if we let $u = \cos(x)$, so that $\sin(x)\,dx$ is replaced by $-du$ by the Substitution Rule, we obtain

$$\begin{aligned}\int \sin^7(x)\cos^3(x)\,dx &= -\int (1-u^2)^3u^3\,du = \int (u^3 - 3u^5 + 3u^7 - u^9)\,du \\ &= \frac{1}{4}u^4 - \frac{1}{2}u^6 + \frac{3}{8}u^8 - \frac{1}{10}u^{10} + c \\ &= \frac{1}{4}\cos^4 - \frac{1}{2}\cos^6 + \frac{3}{8}\cos^8 - \frac{1}{10}\cos^{10} + c\end{aligned}$$

8b. Let $u = 6x$, so that dx is replaced by $\frac{1}{6}du$ by the Substitution Rule and we obtain

$$\int \tan(6x)\,dx = \frac{1}{6}\int \tan(u)\,du.$$

Now use the given formulas for $\int \tan^n(x)\,dx$ and $\int \tan(x)\,dx$:

$$\begin{aligned}\int \tan(6x)\,dx &= \frac{1}{6}\left[\frac{\tan^2(u)}{2} - \int \tan(u)\,du\right] = \frac{1}{6}\left[\frac{\tan^2(u)}{2} - \ln|\sec(u)| + c\right] \\ &= \frac{\tan^2(6x)}{12} - \frac{\ln|\sec(6x)|}{6} + c\end{aligned}$$

8c. We have

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2(\varphi) - 1}\, d\varphi &= \int_{-\pi/3}^{\pi/3} \sqrt{\tan^2(\varphi)}\, d\varphi = \int_{-\pi/3}^{\pi/3} |\tan(\varphi)|\, d\varphi \\ &= \int_{-\pi/3}^{0} [-\tan(\varphi)]\, d\varphi + \int_{0}^{\pi/3} \tan(\varphi)\, d\varphi \\ &= -\big[\ln|\sec(\varphi)|\big]_{-\pi/3}^{0} + \big[\ln|\sec(\varphi)|\big]_{0}^{\pi/3} \\ &= 2\ln(2) = \ln(4).\end{aligned}$$