Math 141 Exam #3 Key (Summer 2011)

1a. Clearly $a_{k+1} = \dfrac{1}{\sqrt{(k+1)^2+4}} \le \dfrac{1}{\sqrt{k^2+4}} = a_k$ for all k, and $\displaystyle\lim_{k\to\infty} \frac{1}{\sqrt{k^2+4}} = 0$, so $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k^2+4}}$ converges by the Alternating Series Test.

1b. Diverges by the Divergence Test since $\displaystyle\lim_{k\to\infty} \left|(-1)^{k+1}\frac{2k^2+3}{5k^2+1}\right| = \lim_{k\to\infty} \frac{2k^2+3}{5k^2+1} = \frac{2}{5} \neq 0$ implies that $\displaystyle\lim_{k\to\infty} (-1)^{k+1}\frac{2k^2+3}{5k^2+1} \neq 0$.

2. We have $\dfrac{1}{(2k+1)^3} < 10^{-3}$ when $(2k+1)^3 > 1000$, which occurs when $2k+1 > 10$. Solving, we arrive at $k > 4.5$, or in our case $k = 5$ since k must be an integer. By the Remainder Theorem, then, the error will be less than 10^{-3} if we estimate the series by $\sum_{k=1}^{4} (-1)^k/(2k+1)^3 = -1/3^3 + 1/5^3 - 1/7^3 + 1/9^3 \approx -0.0306$.

3a. Applying Ratio Test, $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-1)^{k+1}}{5^{k+1}}\cdot\frac{5^k}{(x-1)^k}\right| = \lim_{k\to\infty}\frac{|x-1|}{5} = \frac{|x-1|}{5}$, so series converges if $\dfrac{|x-1|}{5} < 1$, implying $-4 < x < 6$. When $x = 6$, $\displaystyle\lim_{k\to\infty}\left(\frac{x-1}{5}\right)^k = \lim_{k\to\infty}\left(\frac{6-1}{5}\right)^k = \lim_{k\to\infty}(1) = 1 \neq 0$, so series diverges by Divergence Test. When $x = -4$, $\displaystyle\lim_{k\to\infty}\left(\frac{x-1}{5}\right)^k = \lim_{k\to\infty}\left(\frac{-4-1}{5}\right)^k = \lim_{k\to\infty}(-1)^k \neq 0$, so again the series diverges. Interval of convergence is $(-4, 6)$, radius of convergence is 5.

3b. Applying Ratio Test, $\displaystyle\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(2x+3)^{k+1}}{6(k+1)}\cdot\frac{6k}{(2x+3)^k}\right| = \lim_{k\to\infty}\frac{k|2x+3|}{k+1} = |2x+3|$, so series converges if $-1 < 2x+3 < 1$, implying $-2 < x < -1$. When $x = -2$ series becomes $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{6k}$, which converges by the Alternating Series Test. When $x = -1$ series becomes $\displaystyle\sum_{k=1}^{\infty}\frac{1}{6k}$, which diverges since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ diverges. Interval of convergence is $[-2, -1)$, radius of convergence is $\frac{1}{2}$.

4. $g(x) = \displaystyle\sum_{k=0}^{\infty} 2\cdot(4x)^k$, which converges when $|4x| < 1$, so the interval of convergence is $(-\frac{1}{4}, \frac{1}{4})$.

5. Use the geometric series given in the previous problem to get $f(x) = \dfrac{1}{1-(\sqrt{x}-7)} = \dfrac{1}{8-\sqrt{x}}$. Series converges when $|\sqrt{x}-7| < 1$, which solves to give $6 < \sqrt{x} < 8$ and then $36 < x < 64$. So interval of convergence is $(36, 64)$.

6a. $\dfrac{5^0}{0!}x^0 - \dfrac{5^2}{2!}x^2 + \dfrac{5^4}{4!}x^4 - \dfrac{5^6}{6!}x^6 + \cdots = 1 - \dfrac{25}{2}x^2 + \dfrac{625}{24}x^4 - \dfrac{3125}{144}x^6 + \cdots$

6b. $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k 5^{2k}}{(2k)!}x^{2k} = \sum_{k=0}^{\infty}\frac{(-1)^k(5x)^{2k}}{(2k)!}$

6c. Use the Ratio Test to find the interval of convergence $(-\infty, \infty)$.

7. Using the Maclaurin series for $\tan^{-1}$, $\lim_{x\to 0} \dfrac{3\tan^{-1}x - 3x + x^3}{x^5} = \lim_{x\to 0} \dfrac{3\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right) - 3x + x^3}{x^5} =$ $\lim_{x\to 0} \dfrac{\frac{3}{5}x^5 - \frac{3}{7}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{3}{5} - \frac{3}{7}x^2 + \cdots\right) = \frac{3}{5}$

8. This is #10.4.35 in the book, which was actually done in class before the exam. Answer: 0.1498.

9. From $y = t + 2$ we get $t = y - 2$, and then $x = (t+1)^2$ becomes $x = (y-1)^2$. Note that $t \in [-10, 10]$ implies that $y \in [-8, 12]$, so only part of a parabola results.

10. $(-8, -\pi/3)$ and $(8, -4\pi/3)$.

11. The first thing to notice is that any point where $\theta = \pi/2$ will satisfy the equation, which corresponds to the vertical line $x = 0$. Assuming we're not on this line, we have $x \neq 0$ and thus $r \neq 0$, which then implies $\cos\theta = x/r$ and $\sin\theta = y/r$, and so (recalling $r^2 = x^2 + y^2$ and $\sin 2\theta = 2\sin\theta\cos\theta$), we find that $r\cos\theta = \sin(2\theta)$ $\Rightarrow\ x = \dfrac{2xy}{r^2}\ \Rightarrow\ x = \dfrac{2xy}{x^2+y^2}\ \Rightarrow\ 1 = \dfrac{2y}{x^2+y^2}\ \Rightarrow\ x^2 + (y-1)^2 = 1$. This is a circle centered at $(0, 1)$ with radius 1. So, the graph of $r\cos\theta = \sin(2\theta)$ is as pictured.