Math 141 Exam #2 Key (Summer 2011)

1a. $\dfrac{1}{(x-6)(x+4)}$ decomposes into $\dfrac{1/10}{x-6} - \dfrac{1/10}{x+4}$, so $\displaystyle\int \frac{1}{x^2-2x-24}dx = \int \frac{1}{(x-6)(x+4)}dx = \int \left(\frac{1/10}{x-6} - \frac{1/10}{x+4}\right)dx = \frac{1}{10}\ln|x-6| - \frac{1}{10}\ln|x+4| + C.$

1b. Let $u = x+3$, so $x = u-3$ and $dx = du$, and we get $\displaystyle\int \frac{x}{(x+3)^2}dx = \int \frac{u-3}{u^2}du = \int (u^{-1} - 3u^{-2})du = \ln|u| + 3u^{-1} + C = \ln|x+3| + \frac{3}{x+3} + C.$

2a. Making the substitution $u = \pi/x$ along the way, we proceed thusly: $\displaystyle\int_2^\infty \frac{\sin(\pi/x)}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{\sin(\pi/x)}{x^2}dx = \lim_{b\to\infty}\int_{\pi/2}^{\pi/b} \frac{-\sin(u)}{\pi}du = \lim_{b\to\infty}\left[\frac{\cos(u)}{\pi}\right]_{\pi/2}^{\pi/b} = \lim_{b\to\infty}\frac{\cos(\pi/b)}{\pi} = \frac{\cos(0)}{\pi} = \frac{1}{\pi}.$ That is, the improper integral converges to $1/\pi$.

2b. The function $f(x) = 1/\sqrt[4]{x}$ has a vertical asymptote at $x = 0$, so $\displaystyle\int_0^{16} \frac{1}{\sqrt[4]{x}}dx = \lim_{a\to 0^+}\int_a^{16} x^{-1/4}dx = \lim_{a\to 0^+}\left[\frac{4}{3}x^{3/4}\right]_a^{16} = \lim_{a\to 0^+}\frac{4}{3}(16^{3/4} - a^{3/4}) = \frac{4}{3}\cdot 16^{3/4} = \frac{4}{3}\cdot 8 = \frac{32}{3}.$ The integral converges to $32/3$.

3a. $\frac{1}{32}$, $\frac{1}{64}$.

3b. $a_{n+1} = \frac{1}{2}a_n$, with $a_1 = 1$.

3c. $a_n = \dfrac{1}{2^{n-1}}$ for $n \geq 1$

4a. $\displaystyle\lim_{n\to\infty}\frac{2n^{12}}{7n^{12}+4n^5} = \lim_{n\to\infty}\frac{2}{7+4/n^7} = \frac{2}{7+0} = \frac{2}{7}$

4b. First we evaluate $\displaystyle\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty}n^{1/n} = \lim_{n\to\infty}\exp(\ln n^{1/n}) = \exp\left(\lim_{n\to\infty}\ln n^{1/n}\right) = \exp\left(\lim_{n\to\infty}\frac{\ln n}{n}\right) \overset{LR}{=} \exp\left(\lim_{n\to\infty}\frac{1}{n}\right) = \exp(0) = 1.$

Now, consider the subsequence of $\{a_n\}_{n=1}^\infty$ that consists of the even-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^\infty$ with $n_k = 2k$ for $k \geq 1$. Then, using the fact that $\lim_{n\to\infty} n^{1/n} = 1$, we have $\displaystyle\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty}(-1)^{n_k}n_k^{1/n_k} = \lim_{k\to\infty}(-1)^{2k}(2k)^{1/(2k)} = \lim_{k\to\infty}(2k)^{1/(2k)} = 1.$

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^\infty$ with $n_k = 2k-1$ for $k \geq 1$. Then we have $\displaystyle\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty}(-1)^{2k-1}(2k-1)^{1/(2k-1)} = \lim_{k\to\infty}\left[-(2k-1)^{1/(2k-1)}\right] = -1.$

Since $\{a_n\}$ has two subsequences with different limits, the sequence $\{a_n\}$ itself cannot converge. That is, $\{a_n\}$ diverges.

4c. For all $n \geq 1$ we have $-1 \leq \cos n \leq 1$, and thus $-\frac{1}{2^n} \leq \frac{\cos n}{2^n} \leq \frac{1}{2^n}$ for all n. Since $\lim_{n\to\infty} \frac{-1}{2^n} = 0 = \lim_{n\to\infty} \frac{1}{2^n}$, by the Squeeze Theorem we conclude that $\lim_{n\to\infty} \frac{\cos n}{2^n} = 0$.

5. $\sum_{k=2}^{\infty} \frac{5}{2^k} = \sum_{k=0}^{\infty} \frac{5}{2^{k+2}} = \sum_{k=0}^{\infty} \frac{5}{4}\left(\frac{1}{2}\right)^k = \frac{5/4}{1-1/2} = \frac{5}{2}$

6. $s_n = \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)$ $= \frac{1}{3} - \frac{1}{n+3}$, so $\sum_{k=1}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+3}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{n+3}\right) = \frac{1}{3}$.

7a. $\lim_{k\to\infty} \frac{k}{99k+50} = \frac{1}{99} \neq 0$, so series diverges by the Divergence Test.

7b. Since the function $f(x) = \frac{x}{\sqrt{x^2+4}}$ is not actually nonincreasing on $[1,\infty)$, the Integral Test cannot be used. Unfortunately the series is in textbook under instructions to use this test, so it's an error in the book. The fact is, $\lim_{k\to\infty} \frac{k}{\sqrt{k^2+4}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+4/k^2}} = \frac{1}{\sqrt{1+0}} = 1$, so the series diverges by the Divergence Test.

7c. $\lim_{k\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty} \left|\frac{[(k+1)!]^2}{[2(k+1)]!} \cdot \frac{(2k)!}{(k!)^2}\right| = \lim_{k\to\infty} \frac{(k+1)(k+1)}{(2k+1)(2k+2)} = \lim_{k\to\infty} \frac{k+1}{4k+2} = \frac{1}{4} < 1$, so Ratio Test concludes that the series converges.

7d. $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{k^2/2^k} = \lim_{k\to\infty} \frac{k^{2/k}}{2} = 1/2 < 1$, so Root Test concludes that the series converges.

7e. Use the Limit Comparison Test on the series $\sum_{k=2}^{\infty} \frac{k^2-1}{k^3+9}$ and $\sum_{k=2}^{\infty} \frac{1}{k}$, starting the index k at 2 since, technically, the test requires the series involved to consist of *positive* terms. It's known that $\sum_{k=1}^{\infty} 1/k$ diverges, so therefore $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges also. Now, since $\lim_{k\to\infty} \frac{\frac{k^2-1}{k^3+9}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^3-k}{k^3+9} = 1 \in (0,\infty)$, the LCT concludes that $\sum_{k=2}^{\infty} \frac{k^2-1}{k^3+9}$ must diverge. Therefore the original series $\sum_{k=1}^{\infty} \frac{k^2-1}{k^3+9}$ diverges.

7f. For all $k \geq 1$ we have $\frac{k^8}{k^{11}+3} \leq \frac{k^8}{k^{11}} = \frac{1}{k^3}$, and since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a convergent p-series, it follows by the Direct Comparison Test that the series $\sum_{k=1}^{\infty} \frac{k^8}{k^{11}+3}$ converges.