Math 141 Exam #1 Key (Summer 2011)

1. Set $g(x) = y$, so $y = 2x^3 + 5 \Rightarrow x = \sqrt[3]{\dfrac{y-5}{2}}$. Since $g^{-1}(y) = x$ we obtain $g^{-1}(y) = \sqrt[3]{\dfrac{y-5}{2}}$.

2. Here f is one-to-one and differentiable on $(-\pi/2, \pi/2)$, and $f(\pi/4) = 1$. Then by a theorem in section 7.1 we have $(f^{-1})'(1) = \dfrac{1}{f'(\pi/4)} = \dfrac{1}{\sec^2(\pi/4)} = \dfrac{1}{(\sqrt{2})^2} = \dfrac{1}{2}$.

3. $f'(x) = \dfrac{x-3}{x+1} \cdot \dfrac{(x-3)-(x+1)}{(x-3)^2} = \dfrac{-4}{(x-3)(x+1)}$, valid on $\left\{x \;\middle|\; \dfrac{x+1}{x-3} > 0\right\} = \{x \mid x > 3 \text{ or } x < -1\} = (-\infty, -1) \cup (3, \infty)$.

4a. $\dfrac{1}{2}\ln|4x-3| + C$

4b. $-\dfrac{3}{4}e^{-4t} + C$

4c. $\dfrac{1}{5\ln 5}5^{5x}\Big|_0^5 = \dfrac{1}{5\ln 5}(5^{25}-1) = \dfrac{5^{25}-1}{5\ln 5}$

4d. $\dfrac{1}{11}\sec^{-1}\left|\dfrac{x}{11}\right| + C$

5. We have $(\ln\circ f)(x) = \ln(\sqrt{x})^{\tan x} = \tan(x)\ln(\sqrt{x}) = \dfrac{1}{2}\tan(x)\ln(x) \Rightarrow (\ln\circ f)'(x) = \dfrac{1}{2x}\tan(x) + \dfrac{1}{2}\sec^2(x)\ln(x)$, while by the Chain Rule we have $(\ln\circ f)'(x) = \ln'(f(x)) \cdot f'(x) = f'(x)/f(x)$. Therefore we obtain $f'(x)/f(x)$ $= \dfrac{1}{2x}\tan(x) + \dfrac{1}{2}\sec^2(x)\ln(x) \Rightarrow f'(x) = (\sqrt{x})^{\tan x}\left[\dfrac{\tan(x)}{2x} + \dfrac{\sec^2(x)\ln(x)}{2}\right]$.

6a. $s'(t) = -\sin(3^t) \cdot 3^t \cdot \ln 3$

6b. $g'(x) = \dfrac{4}{(x^2-1)\ln 7} \cdot 2x = \dfrac{8x}{(x^2-1)\ln 7}$

6c. $h'(w) = -\sin(\sin^{-1}(2w)) \cdot \dfrac{1}{\sqrt{1-(2w)^2}} \cdot 2 = -2w \cdot \dfrac{2}{\sqrt{1-(2w)^2}} = \dfrac{-4w}{\sqrt{1-4w^2}}$

6d. $f'(z) = -\dfrac{1}{1+(\sqrt{z})^2} \cdot \dfrac{1}{2\sqrt{z}} = \dfrac{-1}{2\sqrt{z}(1+z)}$.

7. Evaluating: $\lim\limits_{\theta\to\pi/2^-}(\tan\theta)^{\cos\theta} = \lim\limits_{\theta\to\pi/2^-}\exp\left(\ln(\tan\theta)^{\cos\theta}\right) = \lim\limits_{\theta\to\pi/2^-}\exp\left(\dfrac{\ln(\tan\theta)}{\sec\theta}\right) = \exp\left(\lim\limits_{\theta\to\pi/2^-}\dfrac{\ln(\tan\theta)}{\sec\theta}\right)$ $= \exp\left(\lim\limits_{\theta\to\pi/2^-}\dfrac{\cot\theta\sec^2\theta}{\sec\theta\tan\theta}\right) = \exp\left(\lim\limits_{\theta\to\pi/2^-}\dfrac{\sec\theta}{\tan^2\theta}\right) = \exp\left(\lim\limits_{\theta\to\pi/2^-}\dfrac{\cos\theta}{\sin^2\theta}\right) = \exp(0) = 1$

8. Recall that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, so that $\lim_{x\to 0} \ln\left(\dfrac{\sin x}{x}\right) = \ln(1) = 0$ since $\ln(x)$ is continuous at 1, and thus $\lim_{x\to 0}[\ln(\sin x) - \ln x] = 0$. Now, $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2} = \exp\left[\lim_{x\to 0} \ln\left(\dfrac{\sin x}{x}\right)^{1/x^2}\right]$, where we have $\lim_{x\to 0} \ln\left(\dfrac{\sin x}{x}\right)^{1/x^2} = \lim_{x\to 0} \dfrac{\ln(\sin x) - \ln x}{x^2} \overset{LR}{=} \lim_{x\to 0} \dfrac{\cot x - 1/x}{2x} = \lim_{x\to 0} \dfrac{x\cos x - \sin x}{2x^2 \sin x} \overset{LR}{=} \lim_{x\to 0} \dfrac{-x\sin x}{4x\sin x + 2x^2\cos x} = \lim_{x\to 0} \dfrac{-\frac{\sin x}{x}}{\frac{4\sin x}{x} + 2\cos x}$ $= \frac{-1}{4+2} = -\frac{1}{6}$. Therefore $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

9a. $\dfrac{e^{4x}}{32}(8x^2 - 4x + 1) + C$

9b. $-\frac{1}{2}$

9c. $\sec x + 2\cos x - \dfrac{\cos^3 x}{3} + C$

9d. $\ln 4$

9e. $\dfrac{1}{2}\ln\left|\dfrac{\sqrt{4+x^2}}{2} + \dfrac{x}{2}\right| + C$

9f. $8\sin^{-1}\left(\dfrac{x}{4}\right) - \dfrac{x}{2}\sqrt{16 - x^2} + C$