Math 141 Exam #3 Key (Spring 2022)

1a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\sin\frac{\pi}{n} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\cos(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\cos\frac{\pi}{n} = \pi\cos 0 = \pi.$$

1b The limit does not exist, and so the sequence diverges:

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^4-2n}-n^2\right) &= \lim_{n\to\infty}\frac{(\sqrt{n^4-2n}-2n^2)(\sqrt{n^4-2n}+2n^2)}{\sqrt{n^4-2n}+2n^2} = \lim_{n\to\infty} -\frac{2n+3n^4}{\sqrt{n^4-2n}+2n^2}\\ &= \lim_{n\to\infty} -\frac{2/n+3n^2}{\sqrt{1-2/n^3}+2} = \frac{0+\infty}{\sqrt{1-0}+2} = \infty.\end{aligned}$$

1c Take the limit of the logarithm of the function and use L'Hôpital's rule:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \exp\left(2\lim_{n\to\infty}\frac{\ln(1+4/n)}{1/n}\right) \overset{\text{LR}}{=} \exp\left(2\lim_{n\to\infty}\frac{1/(1+4/n)\cdot(-4/n^2)}{-1/n^2}\right)\\ &= \exp\left(2\lim_{n\to\infty}\frac{4}{1+4/n}\right) = \exp(8) = e^8.\end{aligned}$$

2 Reindex to obtain

$$\sum_{n=0}^{\infty}\frac{2}{5^{n+3}} = \sum_{n=0}^{\infty}\frac{2}{125}\left(\frac{1}{5}\right)^n = \frac{2/125}{1-1/5} = \frac{1}{50}.$$

3 The nth partial sum is

$$\begin{aligned}s_n &= (\ln 3 - \ln 1) + (\ln 4 - \ln 2) + \cdots + [\ln(n+1) - \ln(n-1)] + [\ln(n+2) - \ln n]\\ &= -\ln 1 - \ln 2 + \ln(n+1) + \ln(n+2) = \ln(n+1)(n+2) - \ln 2,\end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[\ln(n+1)(n+2) - \ln 2\right] = \infty.$$

That is, the series diverges.

4 Find the smallest integer value of n for which $\frac{1}{10n^4} < \frac{1}{10{,}000}$. Since

$$\frac{1}{10n^4} < \frac{1}{10,000} \quad\Rightarrow\quad n^4 > 1000,$$

and 6 is the first integer for which $6^4 > 1000$, estimation with the first five terms will suffice:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{10n^4} \approx \sum_{n=1}^{5} \frac{(-1)^n}{10n^4} = -\frac{1}{10} + \frac{1}{160} - \frac{1}{810} + \frac{1}{2560} - \frac{1}{6250}$$

has absolute error less than 10^{-4}.

5a For all $n \geq 1$ we have

$$0 < \frac{4}{2+3^n n} \leq \frac{4}{3^n n} \leq \frac{4}{3^n},$$

and since $\sum 4/3^n$ is a convergent geometric series, we conclude by the Direct Comparison Test that the given series converges.

5b Since

$$\lim_{n\to\infty} \frac{4^n}{n^2} = +\infty,$$

the series diverges by the Divergence Test.

5c For all $n \geq 1$ we have

$$0 \leq \frac{\tan^{-1} n}{n^2} \leq \frac{\pi}{2n^2},$$

and since $\sum 1/n^2$ is a convergent p-series, it follows that $\sum \pi/2n^2$ is likewise convergent, and therefore the given series converges by the Direct Comparison Test.

5d Since

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} \right| = 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n\to\infty} \exp\left(n \cdot \ln \frac{n}{n+1} \right) \\
&= 2\exp\left(\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n} \right) \overset{\text{LR}}{=} 2\exp\left(\frac{1/n - 1/(n+1)}{-1/n^2} \right) \\
&= 2\exp\left(-\lim_{n\to\infty} \frac{n}{n+1} \right) = 2\exp(-1) = \frac{2}{e} < 1,
\end{aligned}$$

the series converges by the Ratio Test.

5e Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{\ln n}{n} \right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

5f Since

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdots[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{1\cdot 3\cdot 5\cdots(2n-1)}\right)$$
$$= \lim_{n\to\infty}\frac{2(n+1)-1}{2n(2n+1)} = \lim_{n\to\infty}\frac{2n+1}{4n^2+2n} = 0,$$

the series converges by the Ratio Test.

6a Since $(1/n^{5/4})$ is a decreasing sequence of nonnegative values such that $1/n^{5/4} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{5/4}$ is a convergent p-series, the given series is also absolutely convergent.

6b Let Σ represent the given series

$$\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}.$$

Since $n \gg \ln n$ (we've encountered a theorem that states this), there is some N such that $n > \ln n$ for all $n > N$, and so $\frac{1}{n} < \frac{1}{\ln n}$ for $n > N$. Now, since the harmonic series $\sum \frac{1}{n}$ is known to diverge, by the Direct Comparison Test the series $\sum \frac{1}{\ln n}$ must also diverge. This means the series Σ is not absolutely convergent. However, the sequence $b_n = \frac{1}{\ln n}$ is decreasing with $b_n \to 0$ as $n \to \infty$, and so by the Alternating Series Test the series Σ converges. Therefore Σ is conditionally convergent.