# Math 141 Exam #2 Key (Spring 2022)

**1** Let $u = \sqrt{x^2+1}$, so $du = \dfrac{x}{\sqrt{x^2+1}}dx$ and $x^2 = u^2 - 1$, and the integral becomes

$$\int_1^2 6(u^2-1)\,du = 8.$$

**2** With $u = x$, $v' = e^{-4x}$, so that $u' = 1$, $v = -\frac{1}{4}e^{-4x}$, integration by parts gives:

$$\int_0^{\ln 2} xe^{-4x}dx = \left[-\frac{1}{4}xe^{-4x}\right]_0^{\ln 2} + \frac{1}{4}\int_0^{\ln 2} e^{-4x}dx = -\frac{\ln 2}{4}e^{-4\ln 2} - \frac{1}{4}\left[\frac{1}{4}e^{-4x}\right]_0^{\ln 2} = \frac{15-4\ln 2}{256}.$$

**3** With $u = \theta$, $v' = \sec^2\theta$, so that $u' = 1$, $v = \tan\theta$, integration by parts gives:

$$\theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta - \ln|\sec\theta| + C.$$

**4** Let $u = 2x$, so $dx = \frac{1}{2}du$, and use the given reduction formula twice:

$$\begin{aligned}\int \sin^5 2x\,dx = \frac{1}{2}\int \sin^5 u\,du &= -\frac{\sin^4 u\cos u}{10} + \frac{2}{5}\int \sin^3 u\,du \\ &= -\frac{\sin^4 u\cos u}{10} + \frac{2}{5}\left[-\frac{\sin^2 u\cos u}{3} + \frac{2}{3}\int \sin u\,du\right] \\ &= -\frac{\sin^4 2x\cos 2x}{10} - \frac{2\sin^2 2x\cos 2x}{15} - \frac{4}{15}\cos 2x + C.\end{aligned}$$

**5** Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$, and we get (using another reduction formula):

$$\int_0^{\pi/4}\frac{9\tan^2\theta}{9\sec^2\theta}\cdot 3\sec^2\theta\,d\theta = 3\int_0^{\pi/4}\tan^2\theta\,d\theta = 3\left[\tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4}d\theta\right] = \frac{12-3\pi}{4}.$$

**6** We have

$$L = \int_0^{10}\sqrt{1+(2ax)^2}\,dx.$$

Let $x = \frac{1}{2a}\tan\theta$, so $dx = \frac{1}{2a}\sec^2\theta\,d\theta$ and we get (omitting the limits of integration and using yet another reduction formula):

$$\begin{aligned}\int\sqrt{1+(2ax)^2}\,dx = \frac{1}{2a}\int\sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta &= \frac{1}{2a}\int\sec^3\theta\,d\theta \\ &= \frac{1}{2a}\left(\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int\sec\theta\,d\theta\right) \\ &= \frac{\sec\theta\tan\theta}{4a} + \frac{1}{4a}\ln|\sec\theta+\tan\theta| + C \\ &= \frac{x\sqrt{1+(2ax)^2}}{2} + \frac{1}{4a}\ln\left|\sqrt{1+(2ax)^2}+2ax\right| + C.\end{aligned}$$

This gives us an antiderivative for $\sqrt{1+(2ax)^2}$, which we use to find that

$$L = 5\sqrt{1+400a^2} + \frac{1}{4a}\ln\Big(\sqrt{1+400a^2}+20a\Big).$$

**7a** First,

$$\frac{12}{(2s-1)(s-6)} = \frac{A}{2s-1} + \frac{B}{s-6} = \frac{-24/11}{2s-1} + \frac{12/11}{s-6},$$

so

$$\begin{aligned}\int \frac{12}{(2s-1)(s-6)}\,ds &= \int\left(\frac{-24/11}{2s-1} + \frac{12/11}{s-6}]\right)ds\\ &= -\frac{12}{11}\ln|2s-1| + \frac{12}{11}\ln|s-6| + C = \frac{12}{11}\ln\left|\frac{s-6}{2s-1}\right| + C.\end{aligned}$$

**7b** We have

$$\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},$$

so

$$x-5 = Ax(x+1) + B(x+1) + Cx^2 = (A+C)x^2 + (A+B)x + B,$$

which yields the system of equations

$$\begin{cases} A+C = 0\\ A+B = 1\\ \quad\; B = -5\end{cases}$$

The solution to the system is $(A,B,C) = (6,-5,-6)$, so

$$\int \frac{x-5}{x^2(x+1)}\,dx = \int \frac{6}{x}\,dx - \int \frac{5}{x^2}dx - \int \frac{6}{x+1}\,dx = 6\ln|x| + \frac{5}{x} - 6\ln|x+1| + c.$$

**8a** $\displaystyle\lim_{t\to-\infty}\int_t^0 e^x\,dx = \lim_{t\to-\infty}(1-e^t) = 1.$

**8b** By definition the integral $\int_0^{32}$ equals

$$\int_0^1 \frac{1}{(y-1)^{6/5}}\,dy + \int_1^{32} \frac{1}{(y-1)^{6/5}}\,dy,$$

provided both integrals converge. But

$$\int_0^1 \frac{1}{(y-1)^{6/5}}\,dy = \lim_{t\to1^-}\int_0^t \frac{1}{(y-1)^{6/5}}\,dy = -\lim_{t\to1^-}\left[\frac{5}{\sqrt[5]{t-1}}+5\right] = \infty,$$

so the integral $\int_0^{32}$ diverges.

**9** Let $I$ be the integral. For $x\in[1,\infty)$ we find that

$$\frac{1}{x^5+x^3+1} \le \frac{1}{x^5},$$

and since the integral

$$J = \int_1^\infty \frac{1}{x^5}\, dx = \lim_{t\to\infty} \left[ -\frac{1}{4} x^{-4} \right]_1^t = \frac{1}{4},$$

so $J$ converges, the Comparison Theorem implies that $I$ converges also.