Math 141 Exam #1 Key (Spring 2022)

1 By the Inverse Function Theorem $(f^{-1})'(f(a)) = 1/f'(a)$. By trial-and-error we find that $a = 1$ is such that $f(a) = 2$. Since

$$f'(x) = 3 + 3^{-4},$$

we have

$$(f^{-1})'(2) = (f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{1}{6}.$$

2a $f'(x) = -\dfrac{\sin(\ln x)}{x}$

2b $g'(t) = \dfrac{e^{\sqrt{t}+2}}{2\sqrt{t}}$

2c $h'(x) = \dfrac{d}{dx}(e^{x^2 \ln x}) = e^{x^2 \ln x}(2x \ln x + x) = x^{(x^2)}(2x \ln x + x)$

2d $p'(r) = \dfrac{1}{\sqrt{r^2+1} \cdot \sqrt{\left(\sqrt{r^2+2}\right)^2 - 1}} \cdot \dfrac{d}{dr}\sqrt{r^2+2} = \dfrac{r}{(r^2+2)\sqrt{r^2+1}}.$

2e $\varphi'(x) = \cosh(\tan^{-1} x) \cdot \dfrac{1}{1+x^2}$

3 Let $f(x) = xe^{2x}$ and find $f'(\frac{1}{2})$. Have: $f'(x) = (2x+1)e^{2x}$, so $f'(\frac{1}{2}) = 2e$. Tangent line has point $(\frac{1}{2}, \frac{e}{2})$ and slope $2e$, so equation is $y = 2ex - e/2$.

4a Let $u = s^2 - 1$ to get

$$\frac{1}{2}\int_{15}^{24} \frac{1}{u}\,du = \frac{1}{2}\big[\ln u\big]_{15}^{25} = \frac{1}{2}\ln\frac{8}{5}.$$

4b Let $u = -t^2$ to get

$$-\frac{1}{2}\int_0^{-1} e^u\,du = \frac{1}{2}\int_{-1}^{0} e^u\,du = \frac{1}{2}\big[e^u\big]_{-1}^{0} = \frac{1}{2}\left(1 - \frac{1}{e}\right) = \frac{e-1}{2e}.$$

4c Change of base formula can help, then let $u = \ln x$:

$$\int \frac{\ln x}{x \ln 6}\,dx = \frac{1}{\ln 6}\int \frac{\ln x}{x}\,dx = \frac{1}{\ln 6}\int u\,du = \frac{u^2}{2\ln 6} + C = \frac{\ln^2 x}{2\ln 6} + C.$$

4d Let $u = 2\sqrt{x}$ to get

$$\int e^u\,du = e^u + C = e^{2\sqrt{x}} + C.$$

5 $f'(x) = 0$ implies $e^{-x^2}(1 - 2x^2) = 0$ implies $x = \pm\frac{1}{\sqrt{2}}$, the critical points for f. Since $f'(-1) < 0$, $f'(0) > 0$, and $f'(1) < 0$, the Intermediate Value Theorem implies $f' < 0$ on $(-\infty, -\frac{1}{\sqrt{2}})$, $f' > 0$ on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $f' < 0$ on $(\frac{1}{\sqrt{2}}, \infty)$. By the First Derivative Test f has a minimum at $x = -\frac{1}{\sqrt{2}}$ and a maximum at $x = \frac{1}{\sqrt{2}}$.

6a With L'Hôpital's Rule,

$$\lim_{x\to 0^+} (3x)^{x/2} = \exp\left[\lim_{x\to 0^+} \frac{\ln(3x)}{2/x}\right] \overset{\text{LR}}{=} \lim_{x\to 0^+}\left(-\frac{x}{2}\right) = e^0 = 1.$$

6b With L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty} \exp\left[3x\ln\left(\frac{4x}{4x+5}\right)\right] &= \exp\left[\lim_{x\to\infty} \frac{\ln\left(\dfrac{4x}{4x+5}\right)}{\dfrac{1}{3x}}\right] \\
&\overset{\text{LR}}{=} \exp\left[\lim_{x\to\infty} \frac{\dfrac{1}{x} - \dfrac{4}{4x+5}}{-\dfrac{1}{3x^2}}\right] \\
&= \exp\left[\lim_{x\to\infty} \frac{-15x^2}{4x^2+5x}\right] \\
&= \exp(-15/4) = e^{-15/4}.
\end{aligned}$$