Math 141 Exam #4 Key (Spring 2020)

1a This would be the 2nd-order Taylor polynomial:

$$p_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2.$$

1b $\sqrt{3.88} \approx p_2(3.88) = 1.969775.$

2 For $f(x) = \sqrt{1+x}$ we find that $p_1(x) = 1 + x/2$. By a theorem, certainly for $|x| < 1$, we find that the remainder is $R_1(x)$, where

$$|R_1(x)| \le M \cdot \frac{|x-a|^2}{2!}$$

for some M such that $|f''(\xi)| \le M$ for all ξ between a and x. Let $a = 0$, and fix $x \in [-0.12, 0.14]$. For all ξ between 0 and x we have

$$|f''(\xi)| = \left|-\frac{1}{4}(1+\xi)^{-3/2}\right| = \frac{1}{4(1+\xi)^{3/2}} \le \frac{1}{4(1-0.12)^{3/2}} = 0.3028,$$

so we can let $M = 0.3028$. Therefore a suitable bound on the error term is

$$|R_1(x)| \le \frac{0.3028x^2}{2} \le \frac{0.3028(0.14)^{3/2}}{2} = 0.0030$$

for all $x \in [-0.12, 0.14]$.

3a Ratio Test: for any x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(x+3)^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{n^2(x+3)^n}\right| = |x+3| \lim_{n\to\infty} \frac{n^2+2n+1}{n^3+2n^2} = 0$$

and so the series converges on $(-\infty, \infty)$.

3b Ratio Test: for any x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{6x\sqrt{n}}{\sqrt{n+1}}\right| = 6|x| \lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 6|x|,$$

and so the series converges at least on $(-\frac{1}{6}, \frac{1}{6})$. When $x = \frac{1}{6}$ series becomes $\sum \frac{1}{\sqrt{n}}$, a divergent p-series. When $x = -\frac{1}{6}$ series becomes $\sum \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Interval of convergence is $[-\frac{1}{6}, \frac{1}{6})$.

3c Ratio Test: for any x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)n^2}{3(n+1)^2}\right| = \frac{|x-2|}{3},$$

and so the series converges at least on $(-1, 5)$. When $x = -1$ series becomes $\sum \frac{1}{n^2}$, a convergent p-series. When $x = 5$ series becomes $\sum \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test (or just note that the series is absolutely convergent). Interval of convergence is $[-1, 5]$.

4 Use the given Maclaurin series for $\ln(1+x)$:

$$f(x) = \frac{1}{2}\ln(1-x^2) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-x^2)^n}{n} = -\sum_{n=1}^{\infty}\frac{x^{2n}}{2n}$$

for $-1 < -x^2 \leq 1$, or $|x| < 1$. Interval of convergence is $(-1, 1)$.

5 $1 + \frac{3}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots$.

6 Using given Maclaurin series limit becomes

$$\lim_{x\to 0}\frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) - 1 - x}{x^2 - \frac{x^4}{3} + \frac{x^6}{5} + \cdots} = \lim_{x\to 0}\frac{\frac{1}{2} + \frac{x}{6} + \frac{x^2}{24} + \cdots}{1 - \frac{x^2}{3} + \frac{x^4}{5} + \cdots} = \frac{1}{2}.$$

7 Using the Maclaurin series for the sine function:

$$\int_0^1 \sin\sqrt{x}dx = \int_0^1\left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1/2}}{(2n+1)!}\right]dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n+1)!}\int_0^1 x^{n+1/2}dx\right]$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(n+3/2)} = \sum_{n=0}^{\infty}(-1)^n b_n,$$

where

$$b_n = \frac{1}{(2n+1)!(n+3/2)}.$$

We find the lowest n such that $b_n < 10^{-4}$. This turns out to be $b_3 = \frac{1}{22{,}680}$. Thus we make the approximation

$$\int_0^1 \sin\sqrt{x}dx \approx \sum_{n=0}^{2}(-1)^n b_n = \frac{2}{3} - \frac{1}{15} + \frac{1}{420} = 0.60238.$$

8 Use the identity $1 + \tan^2 t = \sec^2 t$ to get $1 + y^2 = x^2$.

9 In general

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{-6\sin 2t}{8\cos 2t},$$

and so the slope is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = -\frac{3\sin(\pi/3)}{4\cos(\pi/3)} = -\frac{3\sqrt{3}}{4}.$$