Math 141 Exam #3 Key (Spring 2020)

1a Using a law of limits:

$$\lim_{n\to\infty} = \left(\lim_{n\to\infty}\frac{n+3}{5n}\right)\left[\lim_{n\to\infty}\left(2-\frac{9}{n}\right)\right] = \frac{1}{5}\cdot 2 = \frac{2}{5}.$$

1b Rationalize the numerator:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\left(n-\sqrt{n^2-n}\right)\left(n+\sqrt{n^2-n}\right)}{n+\sqrt{n^2-n}} = \lim_{n\to\infty}\frac{n}{n+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1-1/n}} = \frac{1}{2}.$$

1c Use L'Hôpital's Rule:

$$\lim_{n\to\infty} a_n \stackrel{\text{LR}}{=} \lim_{n\to\infty}\frac{1/n}{2/(2n)} = \lim_{n\to\infty}(1) = 1.$$

2a $a_n = a_{n-1} - 6,\ a_1 = -2.$

2b $a_n = -2 - 6(n-1) = -6n + 4$ for $n \geq 1$.

3 Reindex to obtain

$$\sum_{n=0}^{\infty}\frac{3}{4}\left(\frac{1}{4}\right)^n = \frac{3/4}{1-1/4} = 1.$$

4 Partial fraction decomposition gives

$$\frac{6}{n^2+2n} = \frac{3}{n} - \frac{3}{n+2}.$$

The nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{3}{k} - \frac{3}{k+2}\right) = 3 + \frac{3}{2} - \frac{3}{n+1} - \frac{3}{n+2},$$

and so

$$\sum_{n=1}^{\infty}\frac{6}{n^2+2n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{9}{2} - \frac{3}{n+1} - \frac{3}{n+2}\right) = \frac{9}{2}.$$

The series converges.

5 We have

$$2.0\overline{45} = 2 + \frac{45}{10^3} + \frac{45}{10^5} + \frac{45}{10^7} + \cdots = 2 + \sum_{n=0}^{\infty} \frac{45}{10^{2n+3}}$$
$$= 2 + \frac{45}{1000}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n = 2 + \frac{45/1000}{1-1/100} = \frac{45}{22}.$$

6a Letting $u = \ln x$, we find that

$$\int_3^\infty \frac{4}{x\sqrt{\ln x}}\,dx = \int_{\ln 3}^\infty \frac{4}{\sqrt{u}}\,du = \lim_{t\to\infty}\left[8\sqrt{u}\right]_{\ln 3}^t = \lim_{t\to\infty}\left(8\sqrt{t} - 8\sqrt{\ln 3}\right) = \infty,$$

and so the series **diverges** by the Integral Test.

6b Since $3^n \gg n^2$ we have

$$\lim_{n\to\infty}\frac{3^n}{n^2+1} = \infty \neq 0,$$

so the series **diverges** by the Divergence Test.

6c Series is expressible as $\sum \frac{1}{2n+1}$, and since

$$\int_1^\infty \frac{1}{2x+1}\,dx = \frac{1}{2}\lim_{t\to\infty}\left[\ln(2t+1) - \ln 3\right] = \infty,$$

the series **diverges** by the Integral Test.

6d For all $k \geq 1$ we have $(2 + \sin k)/k \geq 1/k$, and since $\sum 1/k$ is a divergent p-series, the given series likewise **diverges** by the Direct Comparison Test.

6e Let $a_n = 2^n/(e^n - 1)$ and $b_n = (2/e)^n$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{2^n}{e^n-1}\cdot\frac{e^n}{2^n}\right) = \lim_{n\to\infty}\frac{1}{1-e^{-n}} = 1 \in (0,\infty)$$

and $\sum b_n = \sum(2/e)^n$ is a convergent geometric series, the Limit Comparison Test implies the given series $\sum a_n$ also **converges**.

6f Since

$$\rho = \lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty}\left|\frac{4^{(j+1)^2}}{(j+1)!}\cdot\frac{j!}{4^{j^2}}\right| = \lim_{j\to\infty}\frac{4^{2j+1}}{j+1} = \infty,$$

noting that $4^{2j} \gg j$, the Ratio Test implies the series **diverges**.

7 Let $a_n = 10/(\ln n)^p$, where $p > 0$ is given. Recall that $x^r \gg (\ln x)^q$ for any $q, r > 0$, which by definition means

$$\lim_{x\to\infty} \frac{x^r}{(\ln x)^q} = \infty. \tag{1}$$

Compare $\sum a_n$ (the given series) to $\sum b_n$ for $b_n = 1/n$. Using (1) with $r = 1$ and $q = p$, we obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 10 \lim_{n\to\infty} \frac{n}{(\ln n)^p} = \infty.$$

Since the series $\sum b_n = \sum 1/n$ is a divergent p-series, the Limit Comparison Test implies that the given series $\sum a_n$ also **diverges** for any $p > 0$.

8a The sequence $b_n = n^{-0.99}$ is clearly a decreasing sequence of positive real numbers with limit 0, so the Alternating Series Test implies the given series converges. Because $\sum n^{-0.99}$ is a divergent p-series, however, the given series is not absolutely convergent, and is therefore **conditionally convergent**.

8b Let

$$b_m = \frac{m^2+1}{3m^4+3}.$$

Clearly $b_m > 0$ for all $m \geq 1$, with $b_m \to 0$ as $m \to \infty$. But is the sequence $(b_m)_{m=1}^{\infty}$ nonincreasing? Let

$$f(x) = \frac{x^2+1}{3x^4+3},$$

so $b_m = f(m)$. Since

$$f'(x) = -\frac{6x^3(x^2+1)}{(3x^4+3)^4} < 0$$

for all $x > 0$, it follows that f is a decreasing function on $(0, \infty)$, and therefore $(b_m)_{m=1}^{\infty}$ is a decreasing sequence. The Alternating Series Test now implies the given series converges.

In fact the given series is **absolutely convergent**, as the series $\sum b_m$ can be shown to be convergent using the Limit Comparison Test: comparing with $\sum 1/m^2$ (a p-series known to converge), we have

$$\lim_{m\to\infty} \frac{\dfrac{m^2+1}{3m^4+3}}{1/m^2} = \lim_{m\to\infty} \frac{m^4+m^2}{3m^4+3} = \frac{1}{3} \in (0, \infty).$$