

MATH 141 EXAM #2 KEY (SPRING 2020)

**1a** Let  $u = x + 6$ , so  $du = dx$  and then the integral becomes

$$3 \int \frac{u-6}{\sqrt{u}} du = 3 \int (u^{1/2} - 6u^{-1/2}) du = 2u^{3/2} - 36u^{1/2} + c = 2(x+6)^{3/2} - 36(x+6)^{1/2} + c.$$

**1b** Let  $u = \sec \theta$ , so  $dx = \sec \theta \tan \theta d\theta$  and the integral becomes

$$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \cdot \sec \theta \tan \theta d\theta &= \int_{\pi/4}^{\pi/3} \tan^2 \theta d\theta \\ &= [\tan \theta]_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3} d\theta = \sqrt{3} - 1 - \frac{\pi}{12}, \end{aligned}$$

using the given reduction formula for  $\int \tan^n \theta d\theta$ .

**1c** Partial fraction decomposition and some algebra gives

$$\frac{2x^2 + 7x + 4}{x(x^2 + 2x + 2)} = \frac{2}{x} + \frac{3}{x^2 + 2x + 2} = \frac{2}{x} + \frac{3}{1^2 + (x+1)^2},$$

so integral becomes

$$\int \left[ \frac{2}{x} + \frac{3}{1^2 + (x+1)^2} \right] dx = 2 \ln |x| + 3 \tan^{-1}(x+1) + c.$$

**1d** Let  $u = \cos t$ , so  $du = -\sin t dt$  and integral becomes

$$\int \frac{(1 - \cos^2 t) \sin t}{\cos^5 t} dt = \int \frac{u^2 - 1}{u^5} du = -\frac{1}{2}u^{-2} + \frac{1}{4}u^{-4} + c = -\frac{\sec^2 t}{2} + \frac{\sec^4 t}{4} + c.$$

**1e** Integration by parts: Let  $u = \tan^{-1} 7x$ ,  $v' = x$ , so  $u' = \frac{7}{49x^2+1}$ ,  $v = \frac{1}{2}x^2$ . Then integral becomes

$$\frac{1}{2}x^2 \tan^{-1} 7x - \frac{7}{2} \int \frac{x^2}{49x^2 + 1} dx.$$

Do long division on the new integrand to get

$$\frac{x^2}{2} \tan^{-1} 7x - \frac{1}{14} \int \left[ 1 - \frac{7}{(7x)^2 + 1} \right] dx = \frac{x^2}{2} \tan^{-1} 7x - \frac{x}{14} + \frac{1}{98} \tan^{-1} 7x + c.$$

**1f** Integration by parts: Let  $u = \ln x$ ,  $v' = x^2$ , so  $u' = 1/x$ ,  $v = x^3/3$ . Integral becomes

$$\left[ \frac{1}{3}x^3 \ln x \right]_1^e - \frac{1}{3} \int_1^e x^2 dx = \frac{e^3}{3} - \frac{e^3 - 1}{9} = \frac{2e^3 + 1}{9}.$$

**1g** With the substitution  $x = 2 \tan \theta$  we get

$$\begin{aligned} \frac{1}{2} \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{\sec^2 \theta}{\sqrt{4+4 \tan^2 \theta}} d\theta = \frac{1}{2} \int \sec \theta d\theta = \frac{1}{2} \ln |\sec \theta + \tan \theta| + c \\ &= \frac{1}{2} \ln \left| \frac{\sqrt{x^2+4} + x}{2} \right| + c = \frac{1}{2} \ln \left( \frac{\sqrt{x^2+4} + x}{2} \right) + c. \end{aligned}$$

Note the absolute value is evaluated in the end, since in general

$$\sqrt{x^2+4} > \sqrt{x^2} = |x| = -x,$$

and so  $\sqrt{x^2+4} + x > 0$  holds.

**1h** With long division:

$$\frac{x^4+1}{x^3+9x} = x + \frac{1-9x^2}{x^3+9x}.$$

Partial fraction decomposition:

$$x + \frac{1-9x^2}{x^3+9x} = x + \frac{1}{x} - \frac{\frac{82}{9}x}{x^2+9}.$$

Now integral becomes:

$$\int \left( x + \frac{1}{9x} - \frac{82x}{9(x^2+9)} \right) dx = \frac{1}{2}x^2 + \frac{1}{9} \ln |x| - \frac{41}{9} \ln(x^2+9) + c.$$

**2** Volume is

$$\begin{aligned} \int_1^2 \pi \left( \frac{1}{\sqrt{x(3-x)}} \right)^2 dx &= \pi \int_1^2 \frac{1}{x(3-x)} dx = \pi \int_1^2 \left( \frac{1}{3x} + \frac{1}{3(3-x)} \right) dx \\ &= \frac{\pi}{3} \int_1^2 \left( \frac{1}{x} - \frac{1}{x-3} \right) dx = \frac{\pi}{3} [\ln |x| - \ln |x-3|]_1^2 = \frac{\pi}{3} \ln 4. \end{aligned}$$

**3a** Applying partial fraction decomposition, we find the integral converges:

$$\lim_{t \rightarrow \infty} \int_1^t \left( \frac{1}{v} - \frac{1}{v+1} \right) dv = \lim_{t \rightarrow \infty} [\ln |v| - \ln |v+1|]_1^t = \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{t}{t+1} \right| - \ln \frac{1}{2} \right] = \ln 2.$$

**3b** The integral converges:

$$\lim_{t \rightarrow -3^+} \int_t^1 (2y+6)^{-2/3} dy = \lim_{t \rightarrow -3^+} \left[ \frac{3}{2} (2y+6)^{1/3} \right]_t^1 = \frac{3}{2} \lim_{t \rightarrow -3^+} [2 - (2t+6)^{1/3}] = 3.$$

**3c** First, note that

$$\int_0^\infty \frac{e^x}{\sqrt{e^{2x}-1}} dx = \int_0^1 \frac{e^x}{\sqrt{e^{2x}-1}} dx + \int_1^\infty \frac{e^x}{\sqrt{e^{2x}-1}} dx,$$

with both integrals being improper. Working with the second integral, let  $u = e^x$ , so  $du = e^x dx$  and the integral becomes

$$\lim_{t \rightarrow \infty} \int_1^t \frac{e^x}{\sqrt{e^{2x} - 1}} dx = \lim_{t \rightarrow \infty} \int_e^{e^t} \frac{1}{\sqrt{u^2 - 1}} du = \int_e^{\infty} \frac{1}{\sqrt{u^2 - 1}} du.$$

Next let  $u = \sec \theta$  to get

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_e^t \frac{1}{\sqrt{u^2 - 1}} du &= \lim_{t \rightarrow \infty} \int_{\sec^{-1} e}^{\sec^{-1} t} \sec \theta d\theta = \lim_{t \rightarrow \infty} [\ln |\sec \theta + \tan \theta|]_{\sec^{-1} e}^{\sec^{-1} t} \\ &= \lim_{t \rightarrow \infty} [\ln(t + \sqrt{t^2 - 1}) - \ln(e + \sqrt{e^2 - 1})] = \infty. \end{aligned}$$

Since this last integral diverges, so does the original integral  $\int_{-\infty}^{\infty}$ .

**4**  $e^{-ax} = e^{-bx}$  only holds when  $x = 0$ , so the area is

$$\int_0^{\infty} (e^{-bx} - e^{-ax}) dx = \lim_{t \rightarrow \infty} \left[ -\frac{1}{b} e^{-bx} + \frac{1}{a} e^{-ax} \right]_0^t = \frac{1}{b} - \frac{1}{a}.$$

**5** For all  $x \geq 1$  we have

$$\frac{2 + \cos x}{\sqrt{x}} \geq \frac{1}{\sqrt{x}},$$

and since

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} [2\sqrt{x}]_1^t = \lim_{t \rightarrow \infty} (2\sqrt{t} - 2) = \infty,$$

the Comparison Theorem implies that the given integral diverges.