MATH 141 EXAM #1 KEY (SPRING 2020)

1 Since $f(-3) = 12$ and $f'(x) = 2x - 2$, the Inverse Function Theorem gives

$$(f^{-1})'(12) = (f^{-1})'(f(-3)) = \frac{1}{f'(-3)} = \frac{1}{2(-3)-2} = -\frac{1}{8}.$$

2a $\cot x$

2b $-e^x \csc^2(e^x)$

2c $\dfrac{d}{dx}\left[e^{(x^2-x)\ln 3}\right] = 3^{x^2-x}(2x-1)\ln 3$

2d $\dfrac{d}{dx}\left[e^{\sin x \cdot \ln(1+x^2)}\right] = (1+x^2)^{\sin x}\left[\ln(1+x^2)\cos x + \dfrac{2x\sin x}{1+x^2}\right]$

2e Let $y = \log_8 |\tan x|$, so we must find y'. We have $8^y = |\tan x|$, and hence $y = \dfrac{\ln|\tan x|}{\ln 8}$. Now,

$$y' = \frac{\sec^2 x}{\ln 8 \tan x}.$$

2f Using the given formula and the Chain Rule: $\dfrac{1}{x\sqrt{1-\ln^2 x}}$.

2g $-\dfrac{e^t \sec^2 e^t}{|\tan e^t|\sqrt{(\tan e^t)^2 - 1}}$.

2h $\frac{1}{2}(\operatorname{sech} 8z)^{-1/2}(-\tanh 8z \operatorname{sech} 8z)\cdot 8 = -4\tanh 8z\sqrt{\operatorname{sech} 8z}$.

3 Here

$$y' = 2^{\sin x}\cdot \cos x \cdot \ln 2,$$

so when $x = \pi$ we have $y' = -\ln 2$. This is the slope of the tangent line through the point $(\pi, 1)$, so the equation is $y = (\pi - x)\ln 2 + 1$.

4a $\displaystyle\int_{-2}^{3} \frac{12}{18-5t}\,dt = -\frac{12}{5}\Big[\ln|18-5t|\Big]_{-2}^{3} = \frac{12}{5}\ln\left(\frac{28}{3}\right).$

4b Let $u = 1 + \cos x$ to get

$$-\int_2^1 \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^2 = \ln 2.$$

4c Let $u = \ln z$:

$$10 \int \frac{\log_2 z}{z}\, dz = 10 \int \frac{\ln z}{z \ln 2}\, dz = \frac{10}{\ln 2} \int u\, du = \frac{10}{\ln 2} \cdot \frac{u^2}{2} + C = \frac{5}{\ln 2}(\ln z)^2 + C.$$

4d Let $u = x^x$, so $du = x^x(1 + \ln x)dx$ and the integral becomes

$$\int_1^4 du = u\Big|_1^4 = 3.$$

4e Let $u = e^x$, so $du = e^x\, dx$ and the integral becomes

$$\int \frac{1}{u^2 + 4}\, du = \frac{1}{2} \tan^{-1} \frac{u}{2} + C = \frac{1}{2} \tan^{-1} \frac{e^x}{2} + C.$$

4f Let $u = \cosh 3y$ to get

$$\frac{1}{3} \int_1^{\cosh 3} u^3\, du = \frac{1}{12} \left[u^4\right]_1^{\cosh 3} = \frac{\cosh^4 3 - 1}{12}.$$

5 First,

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(8e^{2\sqrt{2}y} - \frac{1}{2} + \frac{2}{16^2} e^{-2\sqrt{2}y}\right) = \left(2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16} e^{-\sqrt{2}y}\right)^2,$$

and so arc length is

$$\int_0^{\ln(2)/\sqrt{2}} \left(2\sqrt{2}e^{\sqrt{2}y} + \frac{\sqrt{2}}{16} e^{-\sqrt{2}y}\right) dy = \frac{65}{32}.$$

6a The limit has the form 3^∞, which is not indeterminate at all but rather equals $+\infty$.

6b We have

$$\exp\left[\lim_{x \to 0} \frac{\ln(x + \cos x)}{x}\right] \overset{\text{LR}}{=} \exp\left(\lim_{x \to 0} \frac{1 - \sin x}{x + \cos x}\right) = \exp(1) = e.$$

6c Limit equals

$$\lim_{x \to 0} e^{(b/x)\ln(1 + a^x)} = \exp\left(\lim_{x \to 0} \frac{b \ln(1 + a^x)}{x}\right) \overset{\text{LR}}{=} \exp\left(\lim_{x \to 0} \frac{b \cdot a^x \cdot \ln a}{1 + a^x}\right) = e^{(b/2)\ln a} = a^{b/2}.$$