Math 141 Exam #4 Key (Spring 2019)

1a Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| = |x|\lim_{n\to\infty}\sqrt[3]{\frac{n}{n+1}} = |x|.$$

Series converges if $|x|<1$, so interval of convergence contains $(-1,1)$. Check endpoints.

At $x=-1$: Series becomes $\sum n^{-1/3}$, which is a divergent p-series.

At $x=1$: Series becomes $\sum(-1)^n n^{-1/3}$, which can be shown to converge by the Alternating Series Test.

Therefore the original series has interval of convergence $(-1,1]$.

1b Apply the Ratio Test, using L'Hôpital's Rule to find the limit:

$$\lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}\cdot\frac{2^n\ln n}{(x+2)^n}\right| = \frac{|x+2|}{2}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{\text{LR}}{=} \frac{|x+2|}{2}\lim_{n\to\infty}\frac{n+1}{n} = \frac{|x+2|}{2}.$$

Series converges if $|x+2|<2$, so interval of convergence contains $(-4,0)$. Check endpoints.

At $x=-4$: Series becomes $\sum(-1)^n/\ln n$, which converges by the Alternating Series Test.

At $x=0$: Series becomes $\sum 1/\ln n$, and since $1/\ln n > 1/n$ for all $n\geq 2$, and the series $\sum 1/n$ is a divergent p-series, the Direct Comparison Test implies that the series diverges.

Interval of convergence is therefore $[-4,0)$.

1c Again apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(5x-4)^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(5x-4)^n}\right| = |5x-4|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = |5x-4|.$$

The series converges if $|5x-4|<1$, so interval of convergence contains $\left(\frac{3}{5},1\right)$. Check endpoints.

At $x=\frac{3}{5}$: Series becomes $\sum(-1)^n/n^3$, which converges by the Alternating Series Test.

At $x=1$: Series becomes $\sum 1/n^3$, a convergent p-series.

Interval of convergence is therefore $\left[\frac{3}{5},1\right]$.

2 Using the formula for a convergent geometric series,

$$f(x)=\frac{x^2}{16}\cdot\frac{1}{1-(-x^4/16)}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{2^{4n+4}}.$$

Apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{4n+6}}{2^{4n+8}}\cdot\frac{2^{4n+4}}{(-1)^n x^{4n+2}}\right| = \frac{x^4}{16}.$$

Series converges if $x^4/16<1$, so $(-2,2)$ is contained in the interval of convergence. Since the series diverges at the endpoints, $(-2,2)$ is the interval of convergence.

3 We have

$$\int_0^{0.2} x\ln(1+x^2)\,dx = \int_0^{0.2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^{0.2}x^{n+1}dx$$
$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.2)^{n+2}}{n(n+2)} = \frac{0.2^3}{3} - \frac{0.2^4}{8} + \frac{0.2^5}{15} - \frac{0.2^6}{24} + \cdots.$$

Since $0.2^6/24 < 10^{-5}$, the estimate

$$\int_0^{0.2} x\ln(1+x^2)\,dx \approx \frac{0.2^3}{3} - \frac{0.2^4}{8} + \frac{0.2^5}{15}$$

will have an absolute error less than 10^{-5}.

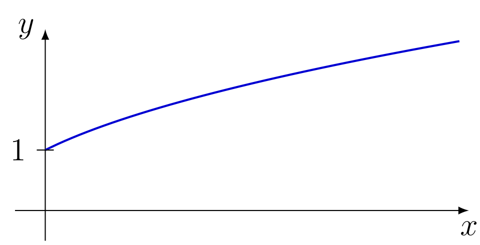
5 Radius of convergence is $R = 1$, with expansion

$$1 - \frac{1}{4}x - \sum_{n=2}^{\infty}\frac{3\cdot 7\cdot\cdots\cdot(4n-5)}{4^n\cdot n!}x^n.$$

6 Length is

$$\int_0^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4}\sec x\,dx = \big[\ln|\sec x + \tan x|\big]_0^{\pi/4} = \ln(\sqrt{2}+1).$$

7 Use the identity $\tan^2 + 1 = \sec^2$ to find that $x + 1 = y^2$, or $y = \sqrt{x+1}$. For $-\pi/2 < t < 0$ travel is from right to left in the curve below, whereupon the curve stops at the point $(0, 1)$ when $t = 0$, and then for $0 < t < \pi/2$ travel is from left to right (and the curve is retraced).



8 We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+1/t}{1-1/t} = \frac{t+1}{t-1},$$

and

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{-2/(t-1)^2}{1-1/t} = -\frac{2t}{(t-1)^3}.$$

The curve is concave up for t values such that $d^2y/dx^2 > 0$, or $t \in (0, 1)$.

9 Let $f(\theta) = 2 + \sin\theta$, so $f'(\theta) = \cos\theta$. Slope is

$$\frac{f'(\pi/4)\sin(\pi/4) + f(\pi/4)\cos(\pi/4)}{f'(\pi/4)\cos(\pi/4) - f(\pi/4)\sin(\pi/4)} = \frac{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + (2 + \frac{1}{\sqrt{2}}) \cdot \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - (2 + \frac{1}{\sqrt{2}}) \cdot \frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{2} - 1.$$

10 One loop is trace for $\theta \in [0, \pi/5]$, so the area is

$$\int_0^{\pi/5} \frac{1}{2}(2\sin 5\theta)^2\, d\theta = \int_0^{\pi/5} (1 - \cos 10\theta)\, d\theta = \frac{\pi}{5} - \frac{1}{10}\sin(2\pi) = \frac{\pi}{5}.$$