Math 141 Exam #3 Key (Spring 2019)

1a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\sin\frac{\pi}{n} = \lim_{n\to\infty}\frac{\tan(\pi/n)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\sec^2(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\sec^2\frac{\pi}{n} = \pi\sec^2 0 = \pi.$$

1b We have

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^4-2n}-n^2\right) &= \lim_{n\to\infty}\frac{(\sqrt{n^4-2n}-n^2)(\sqrt{n^4-2n}+n^2)}{\sqrt{n^4-2n}+n^2} = \lim_{n\to\infty}\frac{-2n}{\sqrt{n^4-2n}+n^2}\\ &= \lim_{n\to\infty}\frac{-2/n}{\sqrt{1-2/n^3}+1} = \frac{0}{\sqrt{1-0}+1} = 0.\end{aligned}$$

2 For the first few values of $n \geq 1$ is appears that the sequence is decreasing, so we conjecture that $a_{n+1} < a_n$ for all $n \geq 1$. Since

$$a_{n+1} < a_n \quad\Leftrightarrow\quad \frac{1-(n+1)}{2+(n+1)} < \frac{1-n}{2+n} \quad\Leftrightarrow\quad -n(n+2) < (1-n)(n+3) \quad\Leftrightarrow\quad 0 < 3$$

holds for any $n \geq 1$, and $0 < 3$ is true, we conclude that $a_{n+1} < a_n$ is true for all $n \geq 1$, and therefore the sequence (a_n) is indeed decreasing. We also find that $a_n \to -1$ as $n \to \infty$, so in fact $a_n \in [-1, 0]$ for all $n \geq 1$, and hence (a_n) is bounded.

3 Reindex to obtain

$$\sum_{n=1}^{\infty}\frac{4}{6^n} = \sum_{n=0}^{\infty}\frac{4}{6^{n+1}} = \frac{4}{6}\sum_{n=0}^{\infty}\left(\frac{1}{6}\right)^n = \frac{2}{3}\cdot\frac{1}{1-1/6} = \frac{4}{5}.$$

4 The nth partial sum is

$$\begin{aligned}s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n]\\ &= -\ln 1 + \ln(n+1) = \ln(n+1),\end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}s_n = \lim_{n\to\infty}\ln(n+1) = \infty.$$

That is, the series diverges.

5 By the Integral Test the series converges if and only if the integral

$$I = \int_1^{\infty}\left(\frac{c}{x} - \frac{1}{x+1}\right)dx$$

converges. Since

$$I = \lim_{t\to\infty} \big[c\ln|x| - \ln|x+1|\big]_1^t = \lim_{t\to\infty}\left[\ln\frac{|x|^c}{|x+1|}\right]_1^t = \lim_{t\to\infty}\ln\left(\frac{t^c}{t+1}\right) + \ln 2,$$

and the limit at right goes to $-\infty$ for $c<1$, $+\infty$ for $c>1$, and $\ln 1 = 0$ for $c=1$, we conclude that the series converges if and only if $c=1$.

6a For all $n \geq 1$ we have

$$0 < \frac{4}{2+3^n n} \leq \frac{4}{3^n n} \leq \frac{4}{3^n},$$

and since $\sum 4/3^n$ is a convergent geometric series, we conclude by the Direct Comparison Test that the given series converges.

6b Since

$$\lim_{n\to\infty}\frac{4^n}{n^2} = +\infty,$$

the series diverges by the Divergence Test.

6c For all $n \geq 1$ we have

$$0 \leq \frac{\tan^{-1} n}{n^2} \leq \frac{\pi}{2n^2},$$

and since $\sum 1/n^2$ is a convergent p-series, it follows that $\sum \pi/2n^2$ is likewise convergent, and therefore the given series converges by the Direct Comparison Test.

6d Since

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{2^n n!}\right| = 2\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = 2\lim_{n\to\infty}\exp\left(n\cdot\ln\frac{n}{n+1}\right)\\ &= 2\exp\left(\lim_{n\to\infty}\frac{\ln n - \ln(n+1)}{1/n}\right) \overset{\text{LR}}{=} 2\exp\left(\frac{1/n - 1/(n+1)}{-1/n^2}\right)\\ &= 2\exp\left(-\lim_{n\to\infty}\frac{n}{n+1}\right) = 2\exp(-1) = \frac{2}{e} < 1,\end{aligned}$$

the series converges by the Ratio Test.

6e Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty}\exp\left(-\frac{\ln n}{n}\right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

6f Since

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{1\cdot 3\cdot 5\cdots [2(n+1)-1]}{[2(n+1)-1]!} \cdot \frac{(2n-1)!}{1\cdot 3\cdot 5\cdots (2n-1)} \right)$$
$$= \lim_{n\to\infty} \frac{2(n+1)-1}{2n(2n+1)} = \lim_{n\to\infty} \frac{2n+1}{4n^2+2n} = 0,$$

the series converges by the Ratio Test.

7a Since $(1/n^{5/4})$ is a decreasing sequence of nonnegative values such that $1/n^{5/4} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{5/4}$ is a convergent p-series, the given series is also absolutely convergent.

7b Since

$$\lim_{n\to\infty} \frac{n}{\ln n} = +\infty$$

the series diverges by the Divergence Test.