

MATH 141 EXAM #2 KEY (SPRING 2019)

**1a** Let  $u = \ln x$ ,  $v' = 1$ , so that  $u' = 1/x$ ,  $v = x$ . Then:

$$\int \ln \sqrt{x} dx = \frac{1}{2} \int \ln x dx = \frac{1}{2} \left( x \ln x - \int dx \right) = \frac{x \ln x - x}{2} + C.$$

**1b** Let  $u = \ln(\cos x)$ ,  $v' = \sin x$ , so  $u' = -\tan x$ ,  $v = -\cos x$ . Then:

$$\left[ -\cos x \ln(\cos x) \right]_0^{\pi/3} - \int_0^{\pi/3} \sin x dx = \frac{\ln 2 - 1}{2}.$$

**2a** Let  $u = \cos 6t$ , so

$$\int (\sin 6t)(1 - \cos^2 6t)^2 (\cos^2 6t) dt = -\frac{1}{6} \int (1 - u^2)^2 u^2 du = -\frac{1}{6} \left( \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} \right) + C,$$

and finally

$$-\frac{1}{42} \cos^7 6t + \frac{1}{15} \cos^5 6t - \frac{1}{18} \cos^3 6t + C.$$

**2b** Let  $u = \tan \theta$ , so

$$\int \tan^2 \theta (\tan^2 \theta + 1) \sec^2 \theta d\theta = \int u^2 (u^2 + 1) du = \frac{u^5}{5} + \frac{u^3}{3} + C = \frac{\tan^5 \theta}{5} + \frac{\tan^3 \theta}{3} + C.$$

**2c** Since  $1 + \cos 2x = 2 \cos^2 x$ , we get

$$\int_0^{\pi/6} \sqrt{2 \cos^2 x} dx = \sqrt{2} \int_0^{\pi/6} \cos x dx = \frac{1}{\sqrt{2}}.$$

**3a** Let  $x = \tan \theta$ , so  $dx = \sec^2 \theta d\theta$ , and with a reduction of order formula we have

$$\int_0^{\pi/4} \cos^2 \theta d\theta = \left[ \frac{\cos \theta \sin \theta}{2} \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} d\theta = \frac{1}{4} + \frac{\pi}{8}.$$

**3b** Let  $t = 4 \sec \theta$ , so  $dt = 4 \sec \theta \tan \theta d\theta$ . Then:

$$\int \frac{4 \sec \theta \tan \theta}{16 \sec^2 \theta \sqrt{16 \sec^2 \theta - 16}} d\theta = \frac{1}{16} \int \cos \theta d\theta = \frac{1}{16} \sin \theta + C = \frac{\sqrt{t^2 - 16}}{16t} + C.$$

**4a** With partial fraction decomposition:

$$\int_1^2 \left( \frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3} \right) dy = \left[ 2 \ln |y| + \frac{9}{5} \ln |y+2| + \frac{1}{5} \ln |y-3| \right]_1^2 = \frac{9}{5} \ln \left( \frac{8}{3} \right).$$

**4b** We have

$$\begin{aligned}\int \left( \frac{2}{x} - \frac{x+1}{x^2+3} \right) dx &= 2 \int \frac{1}{x} dx - \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx \\ &= 2 \ln|x| - \frac{1}{2} \ln(x^2+3) - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{x}{\sqrt{3}} \right) + C.\end{aligned}$$

**4c** Let  $u = \sqrt[3]{x}$ , so  $du = \frac{1}{3}x^{-2/3}dx$ , and hence  $dx = 3u^2 du$ . Integral becomes

$$\int_0^1 \frac{3u^2}{u+1} du = \int_0^1 \left( 3u - 3 + \frac{3}{u+1} \right) du = \ln 8 - \frac{3}{2}.$$

**5a** Let  $u = e^x$ :

$$\int e^x \cdot e^{e^x} dx = \int e^u du = e^u + C = e^{e^x} + C.$$

**5b** Integration by parts: set  $u = \arctan \sqrt{x}$  and  $v' = 1$ , so  $u' = \frac{1}{2(x+1)\sqrt{x}}$  and  $v = x$ .

Integral becomes:

$$x \arctan \sqrt{x} - \int \frac{1}{x+1} \cdot \frac{1}{2\sqrt{x}} dx.$$

Now let  $w = \sqrt{x}$  in the new integral to get

$$x \arctan \sqrt{x} - \int \frac{w^2}{w^2+1} dw = x \arctan \sqrt{x} - \int \left( 1 - \frac{1}{w^2+1} \right) dw,$$

where long division shows that

$$\frac{w^2}{w^2+1} = 1 - \frac{1}{w^2+1}.$$

Finally we obtain

$$(x+1) \arctan \sqrt{x} - \sqrt{x} + C.$$

**5c** Rewrite the radicand as a difference of squares, then make the substitution  $u = x+1$  followed by the substitution  $u = 2 \sin \theta$ :

$$\begin{aligned}\int \sqrt{4 - (x+1)^2} dx &= \int \sqrt{4 - u^2} du = \int \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta d\theta \\ &= 4 \int \cos^2 \theta d\theta = 4 \left( \frac{\cos \theta \sin \theta}{2} + \frac{1}{2} \int d\theta \right) \\ &= 2 \cos \theta \sin \theta + 2\theta + C = 2 \cdot \frac{\sqrt{4-u^2}}{2} \cdot \frac{u}{2} + 2 \sin^{-1} \frac{u}{2} + C \\ &= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2 \sin^{-1} \left( \frac{x+1}{2} \right) + C.\end{aligned}$$

**6a** Let  $u = 1 + x^3$ , so  $du = 3x^2 dx$ , and integral becomes

$$\int_1^\infty \frac{1/3}{\sqrt{u}} du = \lim_{t \rightarrow \infty} \left[ \frac{2}{3} \sqrt{u} \right]_1^t = \lim_{t \rightarrow \infty} \frac{2}{3} (\sqrt{t} - 1) = \infty.$$

Integral diverges.

**6b** Integral converges:

$$\lim_{t \rightarrow -2^+} \left[ \frac{4}{3} (x+2)^{3/4} \right]_t^{14} = \frac{4}{3} \lim_{t \rightarrow -2^+} [8 - (t+2)^{3/4}] = \frac{32}{3}.$$

**6c** Integration by parts yields

$$\left[ \frac{z}{2} e^{2z} \right]_{-\infty}^0 - \int_{-\infty}^0 \frac{1}{2} e^{2z} dz = \lim_{t \rightarrow -\infty} \left[ \frac{z}{2} e^{2z} \right]_t^0 - \frac{1}{4} \lim_{t \rightarrow -\infty} [e^{2z}]_t^0 = -\frac{1}{4}.$$

Integral converges.

**7** For all  $x \in (0, 1]$  we have

$$\frac{\sec^2 x}{x\sqrt{x}} \geq \frac{1}{x\sqrt{x}},$$

and since

$$\int_0^1 \frac{1}{x\sqrt{x}} dx = - \lim_{t \rightarrow 0^+} \left[ \frac{2}{\sqrt{x}} \right]_t^1 = -2 \lim_{t \rightarrow 0^+} \left( 1 - \frac{1}{\sqrt{t}} \right) = \infty,$$

the integral

$$\int_0^1 \frac{\sec^2 x}{x\sqrt{x}} dx$$

diverges by the Comparison Theorem.