Math 141 Exam #3 Key (Spring 2018)

1a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\sin\frac{\pi}{n} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{1/n} \stackrel{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\cos(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\cos\frac{\pi}{n} = \pi\cos 0 = \pi.$$

1b We have

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^2+3n}-n\right) &= \lim_{n\to\infty}\frac{(\sqrt{n^2+3n}-n)(\sqrt{n^2+3n}+n)}{\sqrt{n^2+3n}+n} = \lim_{n\to\infty}\frac{3n}{\sqrt{n^2+3n}+n}\\ &= \lim_{n\to\infty}\frac{3}{\sqrt{1+3/n}+1} = \frac{3}{\sqrt{1+0}+1} = \frac{3}{2}.\end{aligned}$$

2 Since $-\pi/2 < \tan^{-1} n < \pi/2$ for any integer n, we have

$$-\frac{2\pi}{n^4+1} < \frac{4\tan^{-1} n}{n^4} < \frac{2\pi}{n^4+1}$$

for all n, and since

$$\lim_{n\to\infty}\frac{2\pi}{n^4+1} = 0,$$

the Squeeze Theorem implies that

$$\lim_{n\to\infty}\frac{4\tan^{-1} n}{n^4} = 0.$$

3 Reindex to obtain

$$\sum_{n=1}^{\infty}\frac{6}{4^n} = \sum_{n=0}^{\infty}\frac{6}{4^{n+1}} = \frac{6}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{3}{2}\cdot\frac{1}{1-1/4} = 2.$$

4 The nth partial sum is

$$\begin{aligned}s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n]\\ &= -\ln 1 + \ln(n+1) = \ln(n+1),\end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\ln(n+1) = \infty.$$

That is, the series diverges.

5 Find the smallest integer value of n for which $\frac{1}{2n^4} < \frac{1}{1000}$. Since

$$\frac{1}{2n^4} < \frac{1}{1000} \quad\Rightarrow\quad n^4 > 500,$$

and $4^4 < 500$ while $5^4 > 500$, the estimation

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^4} \approx \sum_{n=1}^{4} \frac{(-1)^n}{2n^4} = -\frac{1}{2} + \frac{1}{32} - \frac{1}{162} + \frac{1}{512}$$

has absolute error less than 10^{-3}.

6a For all $n \geq 1$ we have

$$0 < \frac{4}{2 + 3^n n} \leq \frac{4}{3^n n} \leq \frac{4}{3^n},$$

and since $\sum 4/3^n$ is a convergent geometric series, we conclude by the Direct Comparison Test that the given series converges.

6b Since

$$\lim_{n\to\infty} \frac{4^n}{n^2} = +\infty,$$

the series diverges by the Divergence Test.

6c For all $n \geq 1$ we have

$$0 \leq \frac{\tan^{-1} n}{n^2} \leq \frac{\pi}{2n^2},$$

and since $\sum 1/n^2$ is a convergent p-series, it follows that $\sum \pi/2n^2$ is likewise convergent, and therefore the given series converges by the Direct Comparison Test.

6d Since

$$\begin{aligned}
\rho = \lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} \right| &= 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n\to\infty} \exp\left(n \cdot \ln \frac{n}{n+1} \right) \\
&= 2 \exp\left(\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n} \right) \overset{\text{LR}}{=} 2 \exp\left(\frac{1/n - 1/(n+1)}{-1/n^2} \right) \\
&= 2 \exp\left(-\lim_{n\to\infty} \frac{n}{n+1} \right) = 2 \exp(-1) = \frac{2}{e} < 1,
\end{aligned}$$

the series converges by the Ratio Test.

6e Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{\ln n}{n} \right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

6f Since

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{1\cdot 3\cdot 5\cdots[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{1\cdot 3\cdot 5\cdots(2n-1)}\right)\\ &= \lim_{n\to\infty}\frac{2(n+1)-1}{2n(2n+1)} = \lim_{n\to\infty}\frac{2n+1}{4n^2+2n} = 0,\end{aligned}$$

the series converges by the Ratio Test.

7a Since $(1/n^{5/4})$ is a decreasing sequence of nonnegative values such that $1/n^{5/4} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{5/4}$ is a convergent p-series, the given series is also absolutely convergent.

7b Since

$$\lim_{n\to\infty}\frac{n}{\ln n} = +\infty$$

the series diverges by the Divergence Test.