Math 141 Exam #4 Key (Spring 2017)

1 4th-order Taylor polynomial is $p_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$. Now, converting to radians,

$$\cos(2^\circ) = \cos\left(\tfrac{\pi}{90}\right) \approx p_4\left(\tfrac{\pi}{90}\right) = 1 - \frac{1}{2}\left(\frac{\pi}{90}\right)^2 + \frac{1}{24}\left(\frac{\pi}{90}\right)^4 \approx 0.999390827.$$

2a Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{(n+1)^2+3}} \cdot \frac{\sqrt{n^2+3}}{x^n}\right| = |x| \lim_{n\to\infty}\left|\sqrt{\frac{n^2+3}{n^2+2n+4}}\right| = |x|.$$

Series converges if $|x| < 1$, so interval of convergence contains $(-1, 1)$. Check endpoints.

At $x = 1$: series becomes $\sum 1/\sqrt{n^2+3}$, and since

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and the series $\sum 1/n$ is known to diverge, the series $\sum 1/\sqrt{n^2+3}$ diverges by the Direct Comparison Test.

At $x = -1$: series becomes $\sum(-1)^n/\sqrt{n^2+3}$, which can be shown to converge by the Alternating Series Test.

Therefore the original series has interval of convergence $[-1, 1)$.

2b Apply the Root Test:

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|2 + \frac{2}{n}\right||x-1| = |x-1|\lim_{n\to\infty}\left(2 + \frac{2}{n}\right) = 2|x-1|.$$

Series converges if $2|x-1| < 1$, so interval of convergence contains $\left(\frac{1}{2}, \frac{3}{2}\right)$. Check endpoints.

At $x = \frac{3}{2}$: Series becomes

$$\sum\left(2 + \frac{2}{n}\right)^n\left(\frac{1}{2}\right)^n = \sum\left(1 + \frac{1}{n}\right)^n,$$

and since $\left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$, the series diverges by the Divergence Test.

At $x = \frac{1}{2}$: Series becomes

$$\sum\left(2 + \frac{2}{n}\right)^n\left(-\frac{1}{2}\right)^n = \sum(-1)^n\left(1 + \frac{1}{n}\right)^n,$$

which also diverges by the Divergence Test.

Therefore the original series has interval of convergence $\left(\frac{1}{2}, \frac{3}{2}\right)$.

2c Apply Ratio Test, using L'Hôpital's Rule to find the limit:

$$\lim\left|\frac{x^{n+1}\ln(n+1)}{x^n \ln n}\right| = |x|\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\text{LR}}{=} |x|\lim_{n\to\infty}\frac{1/(n+1)}{1/n} = |x|.$$

Thus the series converges if $|x| < 1$, so interval of convergence contains $(-1, 1)$. At the endpoints we obtain either the series $\sum \ln n$ or $\sum (-1)^n \ln n$, both of which diverge by the Divergence Test. Therefore the original series has interval of convergence $(-1, 1)$.

3 This is a geometric series, and so

$$\sum_{n=0}^{\infty} \left(\frac{x^2 - 1}{2} \right)^n = \frac{1}{1 - \frac{x^2-1}{2}} = \frac{2}{3 - x^2}$$

for $x \in \big(-\sqrt{3}, \sqrt{3} \big)$.

4a Using the binomial series,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \binom{-1/3}{n} x^{3n} \\
&= 1 - \frac{1}{3}x^3 + \sum_{n=2}^{\infty} \frac{\big(-\frac{1}{3}\big)\big(-\frac{1}{3} - 1\big)\big(-\frac{1}{3} - 2\big) \cdots \big(-\frac{1}{3} - n + 1\big)}{n!} x^{3n} \\
&= 1 - \frac{1}{3}x^3 + \frac{\big(-\frac{1}{3}\big)\big(-\frac{4}{3}\big)}{2!} x^6 + \frac{\big(-\frac{1}{3}\big)\big(-\frac{4}{3}\big)\big(-\frac{7}{3}\big)}{2!} x^9 + \cdots \\
&= 1 - \frac{1}{3}x^3 + \frac{2}{9}x^6 - \frac{14}{81}x^9 + \cdots .
\end{aligned}$$

4b A couple ways:

$$1 + \sum_{n=1}^{\infty} \frac{\big(-\frac{1}{3}\big)\big(-\frac{1}{3} - 1\big)\big(-\frac{1}{3} - 2\big) \cdots \big(-\frac{1}{3} - n + 1\big)}{n!} x^{3n}$$

or

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1)(4)(7) \cdots (3n - 2)}{3^n \, n!} x^{3n}.$$

4c The standard binomial series on the exam is given to converge for $|x| < 1$, and so the binomial series under consideration converges for $|x^3| < 1$. But this immediately implies convergence if and only if $|x| < 1$. Interval of convergence is therefore $(-1, 1)$, and the radius of convergence is $R = 1$.

5 The series of $\ln(1 + x)$ is given. Now,

$$\int_0^{0.1} \frac{\ln(1 + x)}{x} \, dx = \int_0^{0.1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n-1}}{n} \right) dx = \sum_{n=1}^{\infty} \left(\frac{(-1)^{n+1}}{n} \int_0^{0.1} x^{n-1} \, dx \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\frac{1}{n} x^n \right]_0^{0.1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n^2}.$$

This is an alternating series with $b_n = (0.1)^n/n^2$. We have

$$b_1 = \frac{1}{10}, \quad b_2 = \frac{1}{400}, \quad b_3 = \frac{1}{9000}, \quad b_4 = \frac{1}{160,000} < 10^{-5}.$$

By the Alternating Series Estimation Theorem we are assured that

$$\int_0^{0.1} \frac{\ln(1+x)}{x}\, dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n^2} \approx b_1 - b_2 + b_3 = \frac{1757}{18,000}$$

approximates the series (and hence the integral) with an absolute error of less than 10^{-5}.

6 Use the identity $\tan^2 + 1 = \sec^2$ to find that $y^2 + 1 = \sec^2 t$, so that

$$x = \sec^2 t - 1 = (y^2 + 1) - 1 = y^2.$$

Thus we have $x = y^2$ with domain $y \in (-\infty, \infty)$, recalling that $y = \tan t$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

7 The set-up is thus:

$$(x, y) = \left(1 - \tfrac{1}{3}t\right)(3, -4) + \tfrac{1}{3}t\,(2, 0)$$

for $0 \le t \le 3$. Equivalently we may write

$$(x, y) = \left(-\tfrac{1}{3}t + 3, \tfrac{4}{3}t - 4\right), \quad t \in [0, 3]$$

8 Change to $r \sin\theta = e^{r\cos\theta}$, and hence $y = e^x$.

9 Using the formula

$$\int_\alpha^\beta \frac{f^2(\theta) - g^2(\theta)}{2}\, d\theta$$

with $f(\theta) = 2\sqrt{\sin 2\theta}$ and $g(\theta) = 0$, we find the area to be

$$\int_0^{\pi/2} 2\sin 2\theta\, d\theta = 2.$$