Math 141 Exam #2 Key (Spring 2017)

1 The curves intersect where $x = 0, \pm 2\sqrt{2}$. By symmetry (both curves are given by odd functions) the area bounded in the 3rd quadrant is the same as the area bounded in the 1st quadrant. The total area $\mathcal{A}$ is thus twice the area in the 1st quadrant:

$$\mathcal{A} = 2\int_0^{2\sqrt{2}} \left(\frac{8x}{x^2+1} - \frac{x^3}{x^2+1}\right) dx$$

Long division gives

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1},$$

and so, letting $u = x^2+1$, we have

$$\mathcal{A} = 2\left(\int_0^{2\sqrt{2}} \frac{9x}{x^2+1} dx - \int_0^{2\sqrt{2}} x\,dx\right) = 9\int_1^9 \frac{1}{u} du - 2\int_0^{2\sqrt{2}} x\,dx = 9\ln 9 - 8.$$

2 Let $u = \tan^{-1} y^2$ and $v' = y$, so that $u' = \frac{2y}{y^4+1}$ and $v = \frac{1}{2}y^2$:

$$\begin{aligned}\int_0^{1/\sqrt{2}} y\tan^{-1} y^2\,dy &= \left[\frac{1}{2}y^2 \tan^{-1} y^2\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{y^3}{y^4+1} dy \\ &= \frac{1}{4}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{4}\int_1^{5/4} \frac{1}{t}\,dt \\ &= \frac{1}{4}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{4}\ln\left(\frac{5}{4}\right),\end{aligned}$$

where we make the substitution $t = y^4 + 1$.

3 Let $u = x\sin x$ and $v' = \cos x$, so $u' = \sin x + x\cos x$ and $v = \sin x$. Now, using the identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, we have

$$\int x\sin x\cos x\,dx = x\sin^2 x - \int(\sin^2 x + x\cos x\sin x)dx,$$

and so

$$\begin{aligned}2\int x\sin x\cos x\,dx &= x\sin^2 x - \int \sin^2 x\,dx = x\sin^2 x - \frac{1}{2}\int(1-\cos 2x)dx \\ &= x\sin^2 x - \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c\end{aligned}$$

Finally,

$$\int x\sin x\cos x\,dx = \frac{1}{2}x\sin^2 x + \frac{1}{8}\sin 2x - \frac{1}{4}x + c.$$

Note: the reduction formula on the back of the exam could also be used to determine $\int \sin^2 x\,dx$, though the answer will look a little different.

4 Letting $u = \cos\beta$, so $-du = \sin\beta\,d\beta$, we have

$$\int \sin^3\beta\cos^5\beta\,d\beta = \int \cos^5\beta\sin\beta\,d\beta - \int\cos^7\beta\sin\beta\,d\beta$$

$$
\begin{aligned}
&= -\int u^5\,du + \int u^7\,du = -\frac{1}{6}u^6 + \frac{1}{8}u^8 + c \\
&= -\frac{1}{6}\cos^6\beta + \frac{1}{8}\cos^8\beta + c.
\end{aligned}
$$

5 Set $q = 6\tan\theta$, so $dq = 6\sec^2\theta\,d\theta$. Since $q = 6$ implies $\theta = \pi/4$ and $q = 6\sqrt{3}$ implies $\theta = \pi/3$, we obtain:

$$
\int_{\pi/4}^{\pi/3} \frac{36\tan^2\theta\cdot 6\sec^2\theta}{(36\tan^2\theta + 36)^2}d\theta = \int_{\pi/4}^{\pi/3}\frac{\tan^2}{6\sec^2\theta}d\theta = \frac{1}{6}\int_{\pi/4}^{\pi/3}\sin^2\theta\,d\theta = \frac{\pi + 6 - 3\sqrt{3}}{144}.
$$

(Note: the formula for $\int \sin^n x\,dx$ on the back of the exam helps.)

6a Making the substitution $3\tan\theta = x$, so $dx = 3\sec^2\theta\,d\theta$, we have

$$
\text{Area} = \int_0^4 \frac{1}{\sqrt{x^2+9}}dx = \int_0^{\tan^{-1}(\frac{4}{3})}\sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}(\frac{4}{3})} = \ln 3.
$$

6b We have

$$
\text{Volume} = \int_0^4 \pi[f(x)]^2dx = \int_0^4 \frac{\pi}{x^2+9}dx = \left[\frac{\pi}{3}\tan^{-1}\frac{x}{3}\right]_0^4 = \frac{\pi}{3}\tan^{-1}\frac{4}{3}.
$$

7a We have

$$
\frac{12}{(r-4)(r+3)} = \frac{A}{r-4} + \frac{B}{r+3},
$$

so

$$
12 = A(r+3) + B(r-4) = (A+B)r + (3A - 4B),
$$

which yields the system of equations

$$
\begin{cases} A + B = 0 \\ 3A - 4B = 12 \end{cases}
$$

The solution to the system is $(A, B) = \left(\frac{12}{7}, -\frac{12}{7}\right)$, so

$$
\begin{aligned}
\int \frac{12}{(r-4)(r+3)}dr &= \frac{12}{7}\int\frac{1}{r-4}dr - \frac{12}{7}\int\frac{1}{r+3}dr \\
&= \frac{12}{7}\ln|r-4| - \frac{12}{7}\ln|r+3| + c = \frac{12}{7}\ln\left|\frac{r-4}{r+3}\right| + c.
\end{aligned}
$$

7b We have

$$
\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},
$$

so

$$
x - 5 = Ax(x+1) + B(x+1) + Cx^2 = (A+C)x^2 + (A+B)x + B,
$$

which yields the system of equations

$$\begin{cases} A + C = 0 \\ A + B = 1 \\ \phantom{A+{}} B = -5 \end{cases}$$

The solution to the system is $(A, B, C) = (6, -5, -6)$, so

$$\int \frac{x-5}{x^2(x+1)}\,dx = \int \frac{6}{x}\,dx - \int \frac{5}{x^2}dx - \int \frac{6}{x+1}\,dx = 6\ln|x| + \frac{5}{x} - 6\ln|x+1| + c.$$

8a With partial fraction decomposition we obtain

$$\int_{-\infty}^{-2} \frac{2}{t^2-1}dt = \lim_{a\to-\infty} \int_a^{-2} \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \lim_{a\to-\infty} \left[\ln|t-1| - \ln|t+1|\right]_a^{-2}$$

$$= \lim_{a\to-\infty} \left(\ln 3 - \ln\left|\frac{a-1}{a+1}\right|\right) = \ln 3 - \ln 1 = \ln 3.$$

So the integral converges.

8b First, letting $u = -x^2/2$ so that $du = -x\,dx$,

$$\int_0^{\infty} xe^{-x^2/2}dx = \lim_{t\to\infty} \int_0^t xe^{-x^2/2}dx = -\lim_{t\to\infty} \int_0^{-t^2/2} e^u\,du = -\lim_{t\to\infty} \left[e^{-t^2/2} - 1\right] = 1.$$

Now, since $xe^{-x^2/2}$ is an odd function, we have

$$\int_0^{\infty} xe^{-x^2/2}dx = -1.$$

Therefore

$$\int_{-\infty}^{\infty} xe^{-x^2/2}dx = \int_0^{\infty} xe^{-x^2/2}dx + \int_{-\infty}^{0} xe^{-x^2/2}dx = 1 + (-1) = 0.$$

8c Since $z = 3$ is not in the domain of the integrand, we have

$$\int_1^8 \frac{1}{(z-3)^{4/3}}\,dz = \int_1^3 \frac{1}{(z-3)^{4/3}}\,dz + \int_3^8 \frac{1}{(z-3)^{4/3}}\,dz, \tag{1}$$

provided the two integrals at right converge. Now,

$$\int_1^3 \frac{1}{(z-3)^{4/3}}\,dz = \lim_{t\to3^-} \int_1^t \frac{1}{(z-3)^{4/3}}\,dz = \lim_{t\to3^-} \left[-\frac{3}{(z-3)^{1/3}}\right]_1^t$$

$$= \lim_{t\to3^-} \left[\frac{3}{(-2)^{1/3}} - \frac{3}{(t-3)^{1/3}}\right] = +\infty.$$

Ka-Boom!!! The integral at right in (1) diverges. (Note: evaluating the integral in this problem in the usual way would turn up a real value as the answer, but this would be wrong.)