Math 141 Exam #1 Key (Spring 2017)

1 By the Inverse Function Theorem we have $(f^{-1})'(f(a)) = 1/f'(a)$, which, by switching the roles of f and f^{-1}, gives

$$f'(f^{-1}(a)) = \frac{1}{(f^{-1})'(a)}.$$

We're given that $f^{-1}(4) = 7$ and $(f^{-1})'(4) = \frac{4}{5}$, and so

$$f'(7) = f'(f^{-1}(4)) = \frac{1}{(f^{-1})'(4)} = \frac{5}{4}.$$

2a Using the law $\ln(u/v) = \ln u - \ln v$ helps:

$$f'(x) = \frac{1}{x+1} - \frac{1}{x-1}.$$

2b We have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[e^{\ln(\ln x)^{x^2}}\right] = e^{\ln(\ln x)^{x^2}} \frac{d}{dx}\left[\ln(\ln x)^{x^2}\right] = (\ln x)^{x^2} \frac{d}{dx}\left[x^2 \ln(\ln x)\right] \\ &= (\ln x)^{x^2}\left[2x\ln(\ln x) + \frac{x}{\ln x}\right]. \end{aligned}$$

2c We have

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left[e^{\ln(\tan t)^{\tan t}}\right] = e^{\ln(\tan t)^{\tan t}} \frac{d}{dt}\left[(\tan t)\ln(\tan t)\right] \\ &= (\tan t)^{\tan t}\left[(\sec^2 t)\ln(\tan t) + \tan t \cdot \frac{1}{\tan t} \cdot \sec^2 t\right] \\ &= (\tan t)^{\tan t}\left[\ln(\tan t) + 1\right]\sec^2 t. \end{aligned}$$

2d A formula could be used. Or let $y = r(x)$, so

$$y = \log_7 \sqrt[3]{8x} \quad\Rightarrow\quad 7^y = (8x)^{1/3} \quad\Rightarrow\quad y\ln 7 = \frac{1}{3}\ln(8x) \quad\Rightarrow\quad y = \frac{\ln(8x)}{3\ln 7},$$

and hence

$$r'(x) = y' = \frac{d}{dx}\left[\frac{\ln(8x)}{3\ln 7}\right] = \frac{1}{3x\ln 7}.$$

2e By the Chain Rule:

$$\varphi'(z) = \frac{1}{|\ln z|\sqrt{(\ln z)^2 - 1}} \cdot (\ln z)' = \frac{1}{z|\ln z|\sqrt{(\ln z)^2 - 1}}.$$

2f By the Chain Rule:

$$y' = 4\cosh^3(e^{4x}) \cdot \sinh(e^{4x}) \cdot 4e^{4x} = 16e^{4x}\cosh^3(e^{4x})\sinh(e^{4x})$$

3 Here $y' = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \ln x\right)$. At $x = 1$ we thus have $y' = \sin 1$. So the tangent line has slope $\sin 1$ and contains the point $(1, 1^{\sin 1}) = (1, 1)$. The equation of the line is then

$$y = (\sin 1)x - \sin 1 + 1.$$

4a Let $u = 4e^x + 6$, so $\frac{1}{4}du = e^x\,dx$, and we get

$$\int \frac{e^x}{4e^x + 6}dx = \int \frac{1/4}{u}du = \frac{1}{4}\ln|u| + c = \frac{1}{4}\ln(4e^x + 6) + c.$$

4b Let $u = \ln(\ln x)$, so $du = \frac{1}{x \ln x}dx$, and the integral becomes

$$\int \frac{1}{u}du = \ln|u| + c = \ln\big|\ln(\ln x)\big| + c.$$

4c Let $u = \sin x$, so $du = \cos x\,dx = \frac{1}{\sec x}dx$. Integral becomes

$$\int e^u du = e^u + c = e^{\sin x} + c.$$

4d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1 + \cosh t}dt = \int \frac{1}{1+u}du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

5a Let $u = \ln x$, so $du = \frac{1}{x}dx$. Integral becomes

$$\int_0^{\ln 2e} 3^u du = \left[\frac{3^u}{\ln 3}\right]_0^{\ln 2e} = \frac{3^{\ln 2e} - 1}{\ln 3}.$$

5b Let $u = e^x$, so $du = e^x dx$. Integral becomes

$$\int_{1/\sqrt{3}}^{1} \frac{1}{1+u^2}du = \left[\tan^{-1}(u)\right]_{1/\sqrt{3}}^{1} = \tan^{-1}(1) - \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

6 We have

$$\lim_{x\to 2^+} x^{3/(2-x)} = \lim_{x\to 2^+} \exp\left(\frac{3\ln x}{2-x}\right) = \exp\left(\lim_{x\to 2^+}\frac{3\ln x}{2-x}\right),$$

where the limit at right is *not* of an indeterminate form such as $0/0$ or ∞/∞. Thus we cannot apply L'Hôpital's Rule. Instead, we note that $3\ln x \to 3\ln 2 > 0$ as $x \to 2^+$, whereas $2 - x \to 0^-$. As a result we find that

$$\lim_{x\to 2^+}\frac{3\ln x}{2-x} = -\infty,$$

and therefore
$$\lim_{x\to 2^+} x^{3/(2-x)} = 0.$$