Math 141 Exam #4 Key (Spring 2016)

1 Let $f(x) = \sqrt[3]{x}$. The 3rd-order Taylor polynomial centered at 125 for f is

$$\begin{aligned}P_3(x) &= f(125) + f'(125)(x-125) + \frac{f''(125)}{2}(x-125)^2 + \frac{f'''(125)}{6}(x-125)^3 \\ &= 5 + \frac{1}{3}(125)^{-2/3}(x-125) - \frac{1}{9}(125)^{-5/3}(x-125)^2 + \frac{5}{81}(125)^{-8/3}(x-125)^3 \\ &= 5 + \tfrac{1}{75}(x-125) - \tfrac{1}{28{,}125}(x-125)^2 + \tfrac{1}{6{,}328{,}125}(x-125)^3\end{aligned}$$

and so

$$\sqrt[3]{126} = f(126) \approx P_3(126) = 5 + \tfrac{1}{75} - \tfrac{1}{28{,}125} + \tfrac{1}{6{,}328{,}125} \approx 5.0132979358025.$$

(Note this is *very* close to the actual value of 5.01329793496458....)

2a Clearly the series converges when $x = 0$. Assuming $x \neq 0$, we find that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3x^{4(n+1)}}{(n+1)!}\cdot\frac{n!}{n^3x^{4n}}\right| = \lim_{n\to\infty}\frac{(n+1)^2x^4}{n^3} = 0$$

for all x, and so by the Ratio Test the series converges on $(-\infty,\infty)$. There are no endpoints to consider here.

2b Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^nx^{n+1}}{(n+1)^3}\cdot\frac{n^3}{(-1)^{n-1}x^n}\right| = \lim_{n\to\infty}\frac{n^3|x|}{(n+1)^3} = |x|,$$

by the Ratio Test the series converges if $|x| < 1$, which implies $x \in (-1,1)$.

When $x = 1$ the series becomes

$$\sum\frac{(-1)^{n-1}}{n^3},$$

which converges by the Alternating Series Test. When $x = -1$ the series becomes

$$\sum\frac{(-1)^{2n-1}}{n^3} = -\sum\frac{1}{n^3},$$

which is a convergent p-series. The interval of convergence is therefore $[-1,1]$.

2c Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x-1)^{n+1}}{\sqrt[4]{n+1}}\cdot\frac{\sqrt[4]{n}}{(-2)^n(x-1)^n}\right| = \lim_{n\to\infty}2|x-1|\sqrt[4]{\frac{n}{n+1}} = 2|x-1|,$$

by the Ratio Test the series converges if $2|x-1| < 1$, which implies $x \in \left(\frac{1}{2},\frac{3}{2}\right)$.

When $x = 1/2$ the series becomes

$$\sum\frac{1}{\sqrt[4]{n}} = \sum\frac{1}{n^{1/4}},$$

which is a divergent p-series. When $x = 3/2$ the series becomes

$$\sum \frac{(-1)^n}{\sqrt[4]{n}},$$

which converges by the Alternating Series Test. The interval of convergence is therefore $\left(\frac{1}{2}, \frac{3}{2}\right]$.

3 From the table provided we have

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

for $|x| < 1$. Hence

$$\sum_{n=0}^{\infty} \left(\frac{3}{2x^2+1}\right)^n = \frac{1}{1 - \dfrac{3}{2x^2+1}} = \frac{2x^2+1}{2x^2-2}$$

for

$$\left|\frac{3}{2x^2+1}\right| < 1 \quad \Rightarrow \quad 2x^2+1 > 3 \quad \Rightarrow \quad x^2 > 1 \quad \Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty).$$

Thus there are two intervals of convergence: $(-\infty, -1)$ and $(1, \infty)$. (Note: the series is not a power series.)

4a The first four terms of the Taylor series for f centered at 2 are

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3.$$

Now,

$$f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4},$$

and so the first four terms are

$$\frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3.$$

4b Based on the pattern exhibited by the first five terms, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$$

5 From the table provided we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in (-\infty, \infty)$, and so

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

for all x. Now,

$$\int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}+c$$

for all x and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\int_0^{1/2}e^{-x^2}dx=\int_0^{1/2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^{2n}\right)dx=\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}x^{2n+1}\right]_0^{1/2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{2}\right)^{2n+1}.$$

We have arrived at an alternating series $\sum(-1)^n b_n$ with

$$b_n=\frac{1}{n!(2n+1)}\left(\frac{1}{2}\right)^{2n+1}$$

for $n\geq 0$. Evaluating the first few b_n values,

$$b_0=\tfrac{1}{2},\quad b_1=\tfrac{1}{24},\quad b_2=\tfrac{1}{320},\quad b_3=\tfrac{1}{5376},\quad b_4=\tfrac{1}{110{,}592},\quad b_5=\tfrac{1}{2{,}703{,}360},\quad b_6=\tfrac{1}{76{,}677{,}120},$$

we have $b_6\approx 1.30\times 10^{-8}>10^{-8}$, and $b_7\approx 4.05\times 10^{-10}<10^{-8}$. By the Alternating Series Estimation Theorem the approximation

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}\left(\frac{1}{3}\right)^{2n+1}\approx b_0-b_1+b_2-b_3+b_4-b_5+b_6\approx 0.4612810068$$

will have an absolute error that is less than $b_7<10^{-8}$. Hence the approximation

$$\int_0^{1/3}e^{-x^2}dx\approx\tfrac{1}{2}-\tfrac{1}{24}+\tfrac{1}{320}-\tfrac{1}{5376}+\tfrac{1}{110{,}592}-\tfrac{1}{2{,}703{,}360}+\tfrac{1}{76{,}677{,}120}\approx 0.4612810068$$

has an absolute error less than 10^{-8}.

6 Here $x=\sqrt[5]{t}-2$ implies $t=(x+2)^5$, and so $y=t+1$ gives $y=(x+2)^5+1$. Thus we see that

$$f(x)=(x+2)^5+1,$$

and from $t\in[0,32]$ we see that $\mathrm{Dom}(f)=[-2,0]$.

7 There are many possible parametrizations, but one good one is

$$(x(t),y(t))=(8,2)(1-t)+(-2,-3)t=(8-10t,2-5t)$$

for $t\in[0,1]$.

8 It helps to multiply by r to get

$$r^2=2r\sin\theta+2r\cos\theta.$$

Then, since $x=r\cos\theta$, $y=r\sin\theta$, and $r^2=x^2+y^2$, we obtain

$$x^2+y^2=2y+2x.$$

We can improve on this: from $(x^2 - 2x) + (y^2 - 2y) = 0$ we obtain

$$(x-1)^2 + (y-1)^2 = 2,$$

which is seen to be the equation of a circle centered at $(1, 1)$ with radius $\sqrt{2}$.

9 Using the identity $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, area is

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta \\
&= 2\int_0^{2\pi} d\theta + 2\int_0^{2\pi} \cos\theta\, d\theta + \frac{1}{4}\int_0^{2\pi} (1+\cos 2\theta)\, d\theta \\
&= 4\pi + 2\big[\sin\theta\big]_0^{2\pi} + \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = 4\pi + 0 + \frac{\pi}{2} = \frac{9\pi}{2}.
\end{aligned}$$

Thus concludes the exam.
Hurray.