Math 141 Exam #3 Key (Spring 2016)

1a Recurrence relation:

$$a_{n+1} = (-1)^n(\sqrt{|a_n|}+1)^2, \quad a_1 = 1.$$

1b Explicit formula:

$$a_n = (-1)^{n+1}n^2, \quad n \geq 1.$$

2 We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n+1}+3}{3^n} = \lim_{n\to\infty}\left(3 + \frac{1}{3^{n-1}}\right) = 3.$$

3 Use L'Hôpital's Rule along the way to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \exp(\ln(1/n)^{1/n}) = \exp\left(\lim_{n\to\infty} \frac{\ln(1/n)}{n}\right) = \exp\left(\lim_{n\to\infty} \frac{n(-n^{-2})}{1}\right) = e^0 = 1.$$

4 We have

$$1.\overline{25} = 1 + 0.\overline{25} = 1 + \frac{25}{10^2} + \frac{25}{10^4} + \frac{25}{10^6} + \cdots = 1 + 25\sum_{k=1}^{\infty} \frac{1}{10^{2k}} = 1 + 25\sum_{k=1}^{\infty}\left(\frac{1}{10^2}\right)^k$$

$$= 1 + 25\sum_{k=0}^{\infty} 0.01^{k+1} = 1 + \sum_{k=0}^{\infty} 0.25(0.01)^k = 1 + \frac{0.25}{1-0.01} = 1 + \frac{25}{99} = \frac{124}{99}.$$

5 Partial fraction decomposition:

$$\frac{1}{(k+p)(k+p+1)} = \frac{A}{k+p} + \frac{B}{k+p+1} \quad \Rightarrow \quad 1 = A(k+p+1) + B(k+p),$$

and so $(A+B)k + (Ap+Bp+A) = 1$. This gives $A+B=0$ and $Ap+Bp+A=1$, and finally $A=1$ and $B=-1$. So,

$$s_n = \sum_{k=1}^{n} \frac{1}{(k+p)(k+p+1)} = \sum_{k=1}^{n}\left(\frac{1}{k+p} - \frac{1}{k+p+1}\right)$$

$$= \left(\frac{1}{p+1} - \frac{1}{p+2}\right) + \left(\frac{1}{p+2} - \frac{1}{p+3}\right) + \cdots + \left(\frac{1}{p+n-1} - \frac{1}{p+n}\right) + \left(\frac{1}{p+n} - \frac{1}{p+n+1}\right)$$

$$= \frac{1}{p+1} - \frac{1}{p+n+1}.$$

From this we see that

$$\sum_{k=1}^{\infty} \frac{1}{(k+p)(k+p+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(k+p)(k+p+1)} = \lim_{n\to\infty} s_n = \frac{1}{p+1}.$$

6 We have

$$\sum_{n=1}^{\infty} \frac{n^e}{n^\pi} = \sum_{n=1}^{\infty} \frac{1}{n^{\pi-e}},$$

where $\pi - e \approx 0.42$ shows that $0 < \pi - e < 1$, and so the series is a divergent p-series.

7 Letting $u = \ln x$, we have

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2} dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du$$
$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln t} = \lim_{t\to\infty} \left(\frac{1}{\ln 2} - \frac{1}{\ln t}\right) = \frac{1}{\ln 2}.$$

Since the integral converges, the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ also converges by the Integral Test.

8 We'll use the Limit Comparison Test, comparing the given series with $\sum_{n=1}^\infty \frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2n-\sqrt{n}}} = \lim_{n\to\infty} \left(2 - \frac{1}{\sqrt{n}}\right) = 2 \in (0, \infty),$$

and so since $\sum_{n=1}^\infty \frac{1}{n}$ is known to diverge, the series $\sum_{n=1}^\infty \frac{1}{2n-\sqrt{n}}$ must also diverge.

9 We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{(n+1)^{100}}{(n+2)!} \cdot \frac{(n+1)!}{n^{100}}\right] = \lim_{n\to\infty} \frac{1}{n+2}\left(1 + \frac{1}{n}\right)^{100} = 0 \in [0, 1),$$

and so the series converges by the Ratio Test.

10 Using L'Hôpital's Rule in the end (steps omitted), we have

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{2n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} = \lim_{n\to\infty} e^{2n\ln\left(\frac{n}{n+1}\right)} = e^{-2} \in [0, 1),$$

and so the series converges by the Root Test.

11 Note that

$$\lim_{n\to\infty} \left|(-1)^n \frac{n^2-1}{4n^2+9}\right| = \lim_{n\to\infty} \frac{n^2-1}{4n^2+9} = \frac{1}{4} \neq 0,$$

which shows that

$$\lim_{n\to\infty} (-1)^n \frac{n^2-1}{4n^2+9} \neq 0$$

also. Hence the series diverges by the Divergence Test.