Math 141 Exam #2 Key (Spring 2016)

1 Applying long division along the way:

$$\int \frac{1}{x^{-1}+1}dx = \int \frac{x}{x+1}dx = \int \left(1 - \frac{1}{x+1}\right)dx = x - \ln|x+1| + c.$$

2a Use integration by parts twice:

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx = -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\,dx\right),$$

which gives

$$2\int e^x \sin x\,dx = e^x \sin x - e^x \cos x,$$

and finally

$$\int e^x \sin x\,dx = \frac{e^x(\sin x - \cos x)}{2} + c.$$

2b The volume is given by

$$\mathcal{V} = \int_1^{e^2} \pi[f(x)]^2 dx = \pi \int_1^{e^2} x^2(\ln x)^2\,dx.$$

By integration by parts with $u(x) = (\ln x)^2$ and $v'(x) = x^2$ we have

$$\mathcal{V} = \pi\left(\left[\frac{x^3}{3}(\ln x)^2\right]_1^{e^2} - \frac{2}{3}\int_1^{e^2} x^2 \ln x\,dx\right) = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\int_1^{e^2} x^2 \ln x\,dx.$$

For the last integral again apply integration by parts, this time with $u(x) = \ln x$ and $v'(x) = x^2$, so

$$\int_1^{e^2} x^2 \ln x\,dx = \left[\frac{x^3}{3}\ln x\right]_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx = \frac{5e^6}{9} + \frac{1}{9}.$$

Therefore

$$\mathcal{V} = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\left(\frac{5e^6}{9} + \frac{1}{9}\right) = \frac{2\pi(13e^6 - 1)}{27} \approx 1220.24.$$

(Note the *exact* answer is what is required here.)

3a Let $u = \cos x$, so that $\sin x\,dx = -du$, to get

$$\int \frac{\sin^5 x}{\cos^2 x}dx = -\int \frac{(1-u^2)^2}{u^2}du = \int\left(2 - \frac{1}{u^2} - u^2\right)du = 2u + \frac{1}{u} - \frac{1}{3}u^3 + c$$

$$= 2\cos x + \sec x - \frac{1}{3}\cos^3 x + c.$$

3b Since $\cot^2 = \csc^2 - 1$, we have

$$\int \cot^4 x\,dx = \int (\csc^2 x - 1)\cot^2 x\,dx = \int \cot^2 x \csc^2 x\,dx - \int \cot^2 x\,dx.$$

Let $u = \cot x$ in the first integral, so that $du = -\csc^2 x\,dx$, and we get

$$\int \cot^2 x \csc^2 x\,dx = -\int u^2\,du = -\frac{1}{3}u^3 = -\frac{1}{3}\cot^3 x.$$

Thus, since $(\cot x)' = -\csc^2 x$,

$$\begin{aligned}\int \cot^4 x\,dx &= -\frac{1}{3}\cot^3 x - \int \cot^2 x\,dx = -\frac{1}{3}\cot^3 x - \int (\csc^2 x - 1)\,dx \\ &= -\frac{1}{3}\cot^3 x + \cot x + x + c.\end{aligned}$$

3c Let $u = \tan x$, so $du = \sec^2 x\,dx$, and with the identity $\sec^2 = \tan^2 + 1$ we obtain

$$\int \tan^9 x \sec^4 x\,dx = \int (u^{11} + u^9)\,du = \frac{1}{12}u^{12} + \frac{1}{10}u^{10} + c = \frac{1}{12}\tan^{12} x + \frac{1}{10}\tan^{10} x + c.$$

4a Let $y = \sin\theta$, so $dy = \cos\theta\,d\theta$. Now, $y \in \left[\frac{1}{2}, 1\right]$ implies $\frac{1}{2} \le \sin\theta \le 1$, and thus $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$. We have

$$\begin{aligned}\int_{1/2}^{1} \frac{\sqrt{1-y^2}}{y^2}dy &= \int_{\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^2\theta}\cos\theta\,d\theta = \int_{\pi/6}^{\pi/2} (\csc^2\theta - 1)\,d\theta = -\big[\cot\theta + \theta\big]_{\pi/6}^{\pi/2} \\ &= \left(\cot\frac{\pi}{6} + \frac{\pi}{6}\right) - \left(\cot\frac{\pi}{2} + \frac{\pi}{2}\right) = \sqrt{3} + \frac{\pi}{6} - 0 - \frac{\pi}{2} = \sqrt{3} - \frac{\pi}{3}.\end{aligned}$$

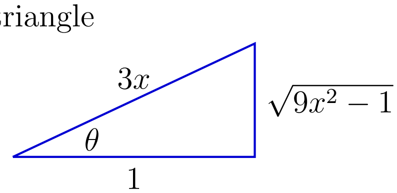
4b Let $x = \frac{1}{3}\sec\theta$ for $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$, so that $dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$. Since $x > \frac{1}{3}$ implies $\sec\theta > 1$ implies $\theta \in [0, \pi/2)$, we have $\tan\theta \ge 0$ and hence

$$\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta.$$

Now,

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = 3\int \frac{\tan\theta}{\sec\theta\sqrt{\tan^2\theta}}\,d\theta = 3\int \cos\theta\,d\theta = 3\sin\theta + c.$$

From $\sec\theta = 3x$ we obtain the triangle



which makes clear that

$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = 3\cdot\frac{\sqrt{9x^2-1}}{3x} + c = \frac{\sqrt{9x^2-1}}{x} + c.$$

5a We have

$$\frac{8}{(y-4)^2(y+3)} = \frac{A}{y-4} + \frac{B}{(y-4)^2} + \frac{C}{y+3},$$

so

$$8 = (A+C)y^2 + (-A+B-8C)y + (-12A+3B+16C),$$

which yields the system of equations

$$\begin{cases} A + C = 0 \\ -A + B - 8C = 0 \\ -12A + 3B + 16C = 8 \end{cases} \tag{1}$$

The solution to the system is $(A, B, C) = \left(-\frac{8}{49}, \frac{8}{7}, \frac{8}{49}\right)$, so

$$\begin{aligned}\int \frac{8}{(y-4)^2(y+3)} &= -\frac{8}{49}\int \frac{1}{y-4}dy + \frac{8}{7}\int \frac{1}{(y-4)^2}dy + \frac{8}{49}\int \frac{1}{y+3}dy \\ &= -\frac{8}{49}\ln|y-4| - \frac{8}{7(y-4)} + \frac{8}{49}\ln|y+3| + c \\ &= \frac{8}{49}\ln\left|\frac{y+3}{y-4}\right| - \frac{8}{7(y-4)} + c.\end{aligned}$$

5b Again start with a decomposition, noting that x^2+2x+6 is an irreducible quadratic:

$$\begin{aligned}\int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2+2x+6}\right)dx \\ &= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{(x+1)^2+5}\,dx. \end{aligned} \tag{2}$$

For the remaining integral, let $u = x+1$ to obtain

$$\int \frac{x+6}{(x+1)^2+5}\,dx = \int \frac{u+5}{u^2+5}\,du = \int \frac{u}{u^2+5}\,du + 5\int \frac{1}{u^2+(\sqrt{5}\,)^2}\,du \tag{3}$$

Letting $w = u^2+5$ in the first integral at right in (3), and using Formula (9) for the second, we next get

$$\begin{aligned}\int \frac{x+6}{(x+1)^2+5}\,dx &= \int \frac{1/2}{w}\,dw + 5\cdot\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln|w| + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2}\ln(u^2+5) + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln[(x+1)^2+5] + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\end{aligned}$$

Returning to (2),

$$\begin{aligned}\int \frac{2}{(x-4)(x^2+2x+6)}\,dx &= \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right] \\ &= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c.\end{aligned}$$

6 We have

$$\lim_{b\to\infty}\int_2^b \frac{1}{(t+2)^2}dt = \lim_{b\to\infty}\left[-\frac{1}{t+2}\right]_2^b = \lim_{b\to\infty}\left(\frac{1}{4}-\frac{1}{b+2}\right) = \frac{1}{4}.$$

7 First, we have

$$\begin{aligned}\int_0^1 \ln(y^2)\,dy &= \lim_{a\to 0^+}\int_a^1 \ln(y^2)\,dy = 2\lim_{a\to 0^+}\int_a^1 \ln(y)\,dy = 2\lim_{a\to 0^+}\big[y\ln y - y\big]_a^1 \\ &= 2\lim_{a\to 0^+}\big[-1-(a\ln a - a)\big] = -2.\end{aligned}$$

By symmetry, then, we also have

$$\int_{-1}^0 \ln(y^2)\,dy = -2.$$

Therefore

$$\int_{-1}^1 \ln(y^2)\,dy = \int_{-1}^0 \ln(y^2)\,dy + \int_0^1 \ln(y^2)\,dy = -4.$$