Math 141 Exam #1 Key (Spring 2016)

1 We have $f'(x) = -2x$, so in particular $f'(2) = -4$. Now, the slope of the graph of f^{-1} at the point $(4, 2)$ is $(f^{-1})'(4)$, where

$$(f^{-1})'(4) = \frac{1}{f'(2)} = -\frac{1}{4}$$

by the Inverse Function Theorem.

2a $f'(x) = -e^x \csc^2(e^x)$.

2b $g'(x) = \dfrac{1}{x \ln x}$.

2c Since $h(x) = e^{\ln(2x+1)^{2x}} = e^{2x\ln(2x+1)}$, we have

$$h'(x) = e^{2x\ln(2x+1)}[2x\ln(2x+1)]' = (2x+1)^{2x}\left[2\ln(2x+1) + \frac{4x}{2x+1}\right].$$

2d $x'(t) = \sec(\sin^{-1} 5t)\tan(\sin^{-1} 5t) \cdot \dfrac{5}{\sqrt{1-(5t)^2}} = \dfrac{5\sec(\sin^{-1} 5t)\tan(\sin^{-1} 5t)}{\sqrt{1-25t^2}}$.

2e $v'(y) = \dfrac{1}{1+\left(\dfrac{1}{y^2+1}\right)^2} \cdot \dfrac{d}{dy}\left(\dfrac{1}{y^2+1}\right) = \dfrac{1}{1+\left(\dfrac{1}{y^2+1}\right)^2} \cdot \dfrac{-2y}{(y^2+1)^2} = -\dfrac{2y}{y^4+2y^2+2}$

2f $\varphi'(x) = 3\tanh^2(x^2)\cdot \operatorname{sech}^2(x^2)\cdot 2x = 6x\tanh^2(x^2)\operatorname{sech}^2(x^2)$.

3a $\displaystyle\int (2e^{-6x} + e^{9x})dx = -\frac{1}{3}e^{-6x} + \frac{1}{9}e^{9x} + c$

3b $\displaystyle\int \frac{9}{4-9y}dy = -\ln|4-9y| + c$

3c Let $u = x^8$, so by the Substitution Rule we replace $x^7\,dx$ with $\frac{1}{8}du$ to get

$$\int x^7 8^{x^8}\,dx = \frac{1}{8}\int 8^u\,du = \frac{1}{8}\cdot\frac{8^u}{\ln 8} + c = \frac{8^{x^8}}{8\ln 8} + c.$$

3d Letting $u = \cosh t$, and noting that $\cosh t > 0$ for all $t \in \mathbb{R}$, we have

$$\int \frac{\sinh t}{1+\cosh t}dt = \int \frac{1}{1+u}du = \ln|u+1| + c = \ln|\cosh t + 1| + c = \ln(\cosh t + 1) + c.$$

4a Let $u = x^x$. Now, since $x^x = e^{\ln x^x} = e^{x\ln x}$, we have

$$\frac{du}{dx} = e^{x\ln x}(\ln x + 1) = (1+\ln x)x^x.$$

Thus we formally have $du = (1+\ln x)x^x\,dx$ when we apply the Substitution Method, giving

$$\int_1^2 (1+\ln x)x^x\,dx = \int_1^4 du = 3.$$

4b Let $u = e^x$, so formally $du = e^x\,dx$ and we have

$$\int_0^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}dx = \int_1^{\sqrt{3}} \frac{1}{1+u^2}du = \left[\tan^{-1}(u)\right]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

5 With LR indicating use of L'Hôpital's Rule, we have

$$\begin{aligned}\lim_{x\to 1^+} \ln(x-1)^{\sin\pi x} &= \lim_{x\to 1^+}\frac{\ln(x-1)}{\csc\pi x} \overset{\text{LR}}{=} \lim_{x\to 1^+}\frac{(x-1)^{-1}}{-\pi\csc\pi x\cot\pi x} = -\frac{1}{\pi}\lim_{x\to 1^+}\frac{\sin\pi x\tan\pi x}{x-1}\\ &\overset{\text{LR}}{=} -\frac{1}{\pi}\lim_{x\to 1^+}\big(\pi\sin\pi x\sec^2\pi x + \pi\cos\pi x\tan\pi x\big) = 0.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{x\to 1^+}\left(\sqrt{x}-1\right)^{2\sin\pi x} &= \lim_{x\to 1^+}(x-1)^{\sin\pi x} = \lim_{x\to 1^+}\exp\left[\ln(x-1)^{\sin\pi x}\right]\\ &= \exp\left[\lim_{x\to 1^+}\ln(x-1)^{\sin\pi x}\right] = \exp(0) = 1.\end{aligned}$$