Math 141 Exam #2 Key (Spring 2015)

1 Applying long division, we have

$$\int_2^4 \frac{x^2+2}{x-1}\,dx = \int_2^4 \left(x+1+\frac{3}{x-1}\right)dx = \left[\frac{x^2}{2}+x+3\ln|x-1|\right]_2^4 = 8+3\ln 3.$$

2a Let $u(x)=x$ and $v'(x)=(x+1)^{-1/2}$, so that $u'(x)=1$ and $v(x)=2(x+1)^{1/2}$, and we have

$$\int \frac{x}{\sqrt{x+1}}dx = 2x\sqrt{x+1} - \int 2\sqrt{x+1}\,dx = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{3/2}+c.$$

2b The volume is given by

$$\mathcal{V} = \int_1^{e^2} \pi[f(x)]^2 dx = \pi\int_1^{e^2} x^2(\ln x)^2\,dx.$$

By integration by parts with $u(x)=(\ln x)^2$ and $v'(x)=x^2$ we have

$$\mathcal{V} = \pi\left(\left[\frac{x^3}{3}(\ln x)^2\right]_1^{e^2} - \frac{2}{3}\int_1^{e^2} x^2\ln x\,dx\right) = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\int_1^{e^2} x^2\ln x\,dx.$$

For the last integral again apply integration by parts, this time with $u(x)=\ln x$ and $v'(x)=x^2$, so

$$\int_1^{e^2} x^2\ln x\,dx = \left[\frac{x^3}{3}\ln x\right]_1^{e^2} - \frac{1}{3}\int_1^{e^2} x^2\,dx = \frac{5e^6}{9}+\frac{1}{9}.$$

Therefore

$$\mathcal{V} = \frac{4\pi e^6}{3} - \frac{2\pi}{3}\left(\frac{5e^6}{9}+\frac{1}{9}\right) = \frac{2\pi(13e^6-1)}{27} \approx 1220.24.$$

(Note the *exact* answer is what is required here.)

3a We have

$$\int (\cos^3 x)\sqrt{\sin x}dx = \int (1-\sin^2 x)\sqrt{\sin x}\cos x\,dx,$$

so if we let $u=\sin x$, so that $\cos x\,dx$ is replaced by du by the Substitution Rule, we obtain

$$\begin{aligned}\int (\cos^3 x)\sqrt{\sin x}dx &= \int (1-u^2)\sqrt{u}\,du = \int (u^{1/2}-u^{5/2})du \\ &= \frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}+c = \frac{2}{3}\sin^{3/2}x - \frac{2}{7}\sin^{7/2}x + c.\end{aligned}$$

3b Let $u = \tan z$, so $du = \sec^2 z\,dz$ and we have

$$\int \frac{\sec^2 z}{\tan^5 z} dz = \int \frac{1}{u^5} du = -\frac{1}{4} u^{-4} + c = -\frac{1}{4\tan^4 z} + c.$$

3c Let $u = e^x + 1$, so $du = e^x\,dx$ and we have

$$\begin{aligned}\int e^x \sec(e^x + 1) dx &= \int \sec u\,du = \ln|\sec u + \tan u| + c \\ &= \ln\big|\sec(e^x + 1) + \tan(e^x + 1)\big| + c.\end{aligned}$$

4a Let $x = \frac{1}{3}\tan\theta$. Formally we obtain $dx = \frac{1}{3}\sec^2\theta d\theta$, and also $\sqrt{9x^2+1} = \sec\theta$. Running through the usual trigonometric substitution process yields

$$\begin{aligned}\int_0^{1/3} \frac{1}{(9x^2+1)^{3/2}} dx &= \frac{1}{3}\int_0^{\pi/4} \frac{\sec^2\theta}{(\tan^2\theta + 1)^{3/2}}\,d\theta = \frac{1}{3}\int_0^{\pi/4} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}} d\theta \\ &= \frac{1}{3}\int_0^{\pi/4} \cos\theta\,d\theta = \frac{1}{3}\Big(\sin\frac{\pi}{4} - \sin 0\Big) = \frac{\sqrt{2}}{6}.\end{aligned}$$

4b Let $t = 13\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that dt is replaced with $13\cos\theta\,d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

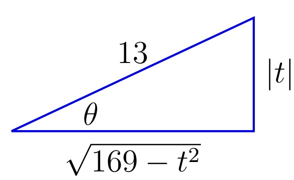
Now,

$$\begin{aligned}\int \sqrt{169 - t^2}\,dt &= \int \sqrt{169 - 169\sin^2\theta} \cdot 13\cos\theta\,d\theta = \int 169\cos\theta\sqrt{1 - \sin^2\theta}\,d\theta \\ &= 169\int \cos\theta\sqrt{\cos^2\theta}\,d\theta = 169\int \cos^2\theta\,d\theta,\end{aligned}$$

and with the deft use of the given formula for $\int \cos^n\theta\,d\theta$ we obtain

$$\int \sqrt{169 - t^2}\,dt = 169\Big(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int d\theta\Big) = \frac{169}{2}\cos\theta\sin\theta + \frac{169}{2}\theta + c.$$

From $t = 13\sin\theta$ comes $\sin\theta = t/13$, so $\theta = \sin^{-1}(t/13)$ and θ may be characterized as an angle in the right triangle



Note that $t \geq 0$ if $\theta \in [0, \pi/2]$, and $t < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{169 - t^2}/13$, and therefore

$$\begin{aligned}\int \sqrt{169 - t^2}\, dt &= \frac{169}{2} \cdot \frac{\sqrt{169 - t^2}}{13} \cdot \frac{t}{13} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c \\ &= \frac{t\sqrt{169 - t^2}}{2} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c.\end{aligned}$$

5a We have

$$\frac{2}{x^3 + x^2} = \frac{2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},$$

whence we obtain

$$2 = Ax(x+1) + B(x+1) + Cx^2 = (A+C)x^2 + (A+B)x + B,$$

which implies we must have $A + C = 0$, $A + B = 0$, and $B = 2$. The only solution is $(A, B, C) = (-2, 2, 2)$. Hence

$$\begin{aligned}\int \frac{2}{x^3 + x^2} dx = \int \left(-\frac{2}{x} + \frac{2}{x^2} + \frac{2}{x+1}\right) dx &= -2\ln|x| - \frac{2}{x} + 2\ln|x+1| + c \\ = \ln\left(\frac{x+1}{x}\right)^2 - \frac{2}{x} + c.\end{aligned}$$

5b Again start with a decomposition, noting that $x^2 + 2x + 6$ is an irreducible quadratic:

$$\begin{aligned}\int \frac{2}{(x-4)(x^2 + 2x + 6)}\, dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2 + 2x + 6}\right) dx \\ &= \frac{1}{15}\ln|x-4| - \frac{1}{15}\int \frac{x+6}{(x+1)^2 + 5}\, dx. \qquad (1)\end{aligned}$$

For the remaining integral, let $u = x + 1$ to obtain

$$\int \frac{x+6}{(x+1)^2 + 5}\, dx = \int \frac{u+5}{u^2 + 5}\, du = \int \frac{u}{u^2 + 5}\, du + 5\int \frac{1}{u^2 + (\sqrt{5}\,)^2}\, du \qquad (2)$$

Letting $w = u^2 + 5$ in the first integral at right in (2), and using Formula (9) for the second, we next get

$$\begin{aligned}\int \frac{x+6}{(x+1)^2 + 5}\, dx &= \int \frac{1/2}{w}\, dw + 5 \cdot \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln|w| + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2}\ln(u^2 + 5) + \sqrt{5}\tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2}\ln[(x+1)^2 + 5] + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\end{aligned}$$

Returning to (1),

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx = \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right]$$

$$= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c.$$

6 The volume of the solid is

$$\mathcal{V} = \int_1^\infty \pi[f(x)]^2 dx = \pi\int_1^\infty \frac{x+1}{x^3}dx = \pi\lim_{b\to\infty}\int_1^b (x^{-1}+x^{-3})\,dx$$

$$= -\pi\lim_{b\to\infty}\left[\frac{1}{x}+\frac{1}{2x^2}\right]_1^b = -\pi\lim_{b\to\infty}\left(\frac{1}{b}+\frac{1}{2b^2}-\frac{3}{2}\right) = \frac{3\pi}{2}.$$

7 The integral is improper since $\ln(0)$ is undefined. Using integration by parts with $u(x) = \ln x$ and $v'(x) = x$ gives

$$\int_0^1 x\ln x\,dx = \lim_{a\to 0^+}\int_a^1 x\ln x\,dx = \lim_{a\to 0^+}\left(\frac{1}{2}\left[x^2\ln x\right]_a^1 - \int_a^1 \frac{x}{2}dx\right)$$

$$= \lim_{a\to 0^+}\left[-\frac{1}{2}a^2\ln a - \frac{1}{4}(1-a^2)\right] = -\frac{1}{4},$$

where by L'Hôpital's Rule we have

$$\lim_{a\to 0^+} a^2\ln a = \lim_{a\to 0^+}\frac{\ln a}{a^{-2}} = \lim_{a\to 0^+}\frac{a^{-1}}{-2a^{-3}} = \lim_{a\to 0^+}\left(-\frac{a^2}{2}\right) = 0.$$