Math 141 Exam #1 Key (Spring 2015)

1 A little trial-and-error readily gives us

$$g(1) = 1^5 - 1^3 + 2(1) = 2,$$

and so the Inverse Function Theorem, along with $g'(x) = 5x^4 - 3x^2 + 2$, implies

$$(g^{-1})'(2) = \frac{1}{g'(1)} = \frac{1}{4}.$$

But can we use the theorem? Note that $g'(1) = 4 > 0$. Since g' is continuous and $g'(1) > 0$, we in fact must have $g'(x) > 0$ for all x in some open interval I containing 1, meaning g is strictly increasing on I, and hence g is one-to-one on I. Therefore $g : I \to g(I)$ has an inverse function $g^{-1} : g(I) \to I$ which, along with the differentiability of g on I, allows us to use the Inverse Function Theorem in the manner above.

2a $f'(x) = \dfrac{2e^{2x}}{e^{2x} + 3}$

2b $\text{Dom}(g) = (0, \infty)$, and for all $x > 0$ we have

$$g(x) = x^{\ln(x^5)} = \exp\left(\ln\left(x^{\ln(x^5)}\right)\right) = \exp\big(\ln(x^5)\ln(x)\big) = \exp\big(5\ln^2(x)\big),$$

and thus

$$g'(x) = \exp\big(5\ln^2(x)\big)\cdot\big(5\ln^2(x)\big)' = x^{\ln(x^5)} \cdot \frac{10\ln(x)}{x} = \frac{10x^{\ln(x^5)}\ln(x)}{x}.$$

2c For x such that $\sin x > 0$ we have

$$h(x) = (\sin x)^{\tan x} = \exp\big(\ln((\sin x)^{\tan x})\big) = \exp(\tan x \cdot \ln(\sin x)),$$

and thus

$$\begin{aligned} h'(x) &= \exp(\tan x \cdot \ln(\sin x))\cdot(\tan x \cdot \ln(\sin x))' \\ &= \exp(\tan x \cdot \ln(\sin x))\cdot\left(\tan x \cdot \frac{\cos x}{\sin x} + \sec^2 x \cdot \ln(\sin x)\right) \\ &= (\sin x)^{\tan x}\left(1 + \ln(\sin x)^{\sec^2 x}\right) \end{aligned}$$

2d $k'(x) = \dfrac{7}{(4 - x^5)\ln(3)} \cdot (4 - x^5)' = -\dfrac{35x^4}{(4 - x^5)\ln(3)}$

2e $\ell'(x) = \dfrac{1}{e^{-2x}\sqrt{(e^{-2x})^2 - 1}} \cdot (e^{-2x})' = \dfrac{-2e^{-2x}}{e^{-2x}\sqrt{e^{-4x} - 1}} = -\dfrac{2}{\sqrt{e^{-4x} - 1}}$

2f $p'(x) = -\dfrac{1}{1+(\sqrt{x}\,)^2} \cdot (\sqrt{x}\,)' = -\dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x}(1+x)}$

3a $\displaystyle\int (3e^{-8x} - 8e^{11x})dx = -\frac{3}{8}e^{-8x} - \frac{8}{11}e^{11x} + c$

3b $\displaystyle\int \frac{9}{4-9y}dy = -\ln|4-9y| + c$

3c Let $u = x^8$, so by the Substitution Rule we replace $x^7\,dx$ with $\frac{1}{8}du$ to get

$$\int x^7 8^{x^8} dx = \frac{1}{8}\int 8^u\,du = \frac{1}{8} \cdot \frac{8^u}{\ln 8} + c = \frac{8^{x^8}}{8\ln 8} + c.$$

4a Let $u = \ln(x)$, so when $x = 1$ we have $u = \ln(1) = 0$, and when $x = 3e$ we have $u = \ln(3e)$. Now, by the Substitution Rule we replace $\frac{1}{x}dx$ with du to get

$$\int_0^{\ln(3e)} \frac{e^u}{2}\,du = \left[\frac{1}{2}e^u\right]_0^{\ln(3e)} = \frac{1}{2}(e^{\ln(3e)} - e^0) = \frac{3e-1}{2}.$$

4b We have

$$5\int_2^{2\sqrt{3}} \frac{1}{z^2+2^2}dz = 5\left[\frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)\right]_2^{2\sqrt{3}} = \frac{5}{2}\left[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\right] = \frac{5}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{5\pi}{24}.$$

5 For x near 0 but not equal to 0, for instance for $x \in I = (-\frac{1}{4}, 0) \cup (0, \frac{1}{4})$, we have

$$\lim_{x\to 0}(x+\cos x)^{1/3x} = \exp\left[\ln(x+\cos x)^{1/3x}\right] = \exp\left[\frac{\ln(x+\cos x)}{3x}\right].$$

The functions $f(x) = \ln(x+\cos x)$ and $g(x) = 3x$ are differentiable on I, with $f(x)/g(x) \to 0/0$ as $x \to 0$. Since

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{1-\sin x}{3(x+\cos x)} = \frac{1}{3},$$

by L'Hôpital's Rule it follows that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\ln(x+\cos x)}{3x} = \frac{1}{3}$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to 0}(x+\cos x)^{1/3x} = \lim_{x\to 0}\exp\left[\frac{\ln(x+\cos x)}{3x}\right] = \exp\left[\lim_{x\to 0}\frac{\ln(x+\cos x)}{3x}\right] = \exp\left(\tfrac{1}{3}\right) = e^{1/3}.$$

6 Using the Chain Rule yields

$$f'(x) = \frac{1}{2}(\tanh 5x)^{-1/2} \cdot \operatorname{sech}^2 5x \cdot 5 = \frac{5\operatorname{sech}^2 5x}{2\sqrt{\tanh 5x}}.$$

7 Make the substitution $u = \sqrt{x}$, and then $w = \cosh u$:

$$\begin{aligned}\int_1^4 \frac{\tanh\sqrt{x}}{\sqrt{x}}\,dx &= \int_1^2 2\tanh u\,du = 2\int_1^2 \frac{\sinh u}{\cosh u}\,du = 2\int_{\cosh 1}^{\cosh 2} \frac{1}{w}\,dw = 2\big[\ln|w|\big]_{\cosh 1}^{\cosh 2} \\ &= 2\ln\left(\frac{\cosh 2}{\cosh 1}\right) = 2\ln\left(\frac{e^2+e^{-1}}{2}\cdot\frac{2}{e+e^{-1}}\right) = 2\ln\left(\frac{e^4+1}{e^4+e^2}\right) \\ &= 2\ln\left(\frac{e^4+1}{e^2+1}\right) - 4.\end{aligned}$$