Math 141 Exam #4 Key (Spring 2014)

1a Applying the Ratio Test,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x+1)^{k+1}}{8^{k+1}}\cdot\frac{8^k}{(x+1)^k}\right| = \lim_{k\to\infty}\frac{|x+1|}{8} = \frac{|x+1|}{8},$$

so the series converges if $|x+1|/8 < 1$, implying $-8 < x+1 < 8$ and thus $-9 < x < 7$. It remains to test the endpoints.

When $x = 7$ the series becomes,

$$\lim_{k\to\infty}\left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty}\left(\frac{7+1}{8}\right)^k = \lim_{k\to\infty}(1) = 1 \neq 0,$$

so the series diverges by the Divergence Test.

When $x = -9$ the series becomes,

$$\lim_{k\to\infty}\left(\frac{x+1}{8}\right)^k = \lim_{k\to\infty}\left(\frac{-9+1}{8}\right)^k = \lim_{k\to\infty}(-1)^k \neq 0,$$

so again the series diverges. Therefore the interval of convergence is $(-9,7)$, and the radius of convergence is $|-9-7|/2 = 8$.

1b Applying the Ratio Test,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(2x+3)^{k+1}}{6(k+1)}\cdot\frac{6k}{(2x+3)^k}\right| = \lim_{k\to\infty}\frac{k|2x+3|}{k+1} = |2x+3|,$$

so the series converges if $-1 < 2x+3 < 1$, implying $-2 < x < -1$.

When $x = -2$ the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{6k},$$

which converges by the Alternating Series Test. When $x = -1$ the series becomes

$$\sum_{k=1}^{\infty}\frac{1}{6k},$$

which diverges since

$$\sum_{k=1}^{\infty}\frac{1}{k}$$

diverges. Interval of convergence is $[-2,-1)$, radius of convergence is $\frac{1}{2}$.

1c Clearly the series converges when $x = -2$. Assuming $x \neq -2$, we can employ the Ratio Test with

$$a_k = (-1)^k\frac{(x+2)^k}{k\cdot 2^k}$$

to obtain

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x+2)^{k+1}}{(k+1)\cdot 2^{k+1}}\cdot\frac{k\cdot 2^k}{(-1)^k(x+2)^k}\right|$$

$$= \lim_{k\to\infty} \left| \frac{(-1)(x+2)}{2(k+1)} \cdot \frac{k}{1} \right| = \lim_{k\to\infty} \frac{k}{2k+2}|x+2| = \frac{1}{2}|x+2|.$$

Thus the series converges if $\frac{1}{2}|x+2| < 1$, which implies $|x+2| < 2$ and thus $-4 < x < 0$. The Ratio Test is inconclusive when $x = -4$ or $x = 0$, so we analyze these endpoint separately.

When $x = -4$ the series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k(-2)^k}{k \cdot 2^k} = \sum_{k=0}^{\infty} \frac{2^k}{k \cdot 2^k} = \sum_{k=0}^{\infty} \frac{1}{k},$$

which is the harmonic series and therefore diverges.

When $x = 0$ the series becomes

$$\sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{k \cdot 2^k} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k},$$

which is an alternating series $\sum(-1)^k b_k$ with $b_k = 1/k$. Since $\lim_{k\to\infty} b_k = 0$ and

$$b_{k+1} = \frac{1}{k+1} < \frac{1}{k} = b_k$$

for all k, by the Alternating Series Test this series converges.

Therefore the series converges on the interval $(-4, 0]$, and the radius of convergence is $R = \frac{1}{2}|0 - (-4)| = 2$.

2 We manipulate to obtain

$$h(x) = 2 \cdot \frac{1}{1-(-3x)} = 2\sum_{k=0}^{\infty}(-3x)^k = \sum_{k=0}^{\infty} 2(-3x)^k,$$

which converges if and only if $|-3x| < 1$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

3 Formally, the function represented by the series is given by

$$f(x) = \frac{1}{1-(\sqrt{x}+4)} = -\frac{1}{3+\sqrt{x}}.$$

The series converges if and only if $|\sqrt{x}+4| < 1$, or equivalently $-5 < \sqrt{x} < -3$. But there exists no $x \in \mathbb{R}$ which satisfies this inequality, and so there is no interval of convergence!

4a We have

$$3x - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \cdots = 3x - \frac{9}{2}x^3 + \frac{625}{24}x^5 - \frac{243}{560}x^7 + \cdots$$

4b $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k(3x)^{2k+1}}{(2k+1)!}$

4c Use the Ratio Test to find that the interval of convergence is $(-\infty, \infty)$.

5 We have

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

for all $x \in (-\infty, \infty)$, and so

$$\sin(x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \tag{1}$$

for all $-\infty < x < \infty$. In particular the series at right in (1) converges on $(-\infty, \infty)$, and so

$$\int \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \right] dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} + c$$

for all $x \in (-\infty, \infty)$ and arbitrary constant c. Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_0^{0.2} \sin(x^2)\, dx &= \int_0^{0.2} \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+2}}{(2k+1)!} \right] dx = \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!} \right]_0^{0.2} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (0.2)^{4k+3}}{(4k+3)(2k+1)!} - \sum_{k=0}^{\infty} \frac{(-1)^k (0)^{4k+3}}{(4k+3)(2k+1)!} \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}
\end{aligned}$$

We have arrived at an alternating series $\sum (-1)^k b_k$ with

$$b_k = \frac{0.2^{4k+3}}{(4k+3)(2k+1)!}$$

for $k \geq 0$. Evaluating the first few b_k values,

$$\begin{aligned}
b_0 &= 0.2^3/(3 \cdot 1!) \approx 2.6667 \times 10^{-3} \\
b_1 &= 0.2^7/(7 \cdot 3!) \approx 3.0476 \times 10^{-7} \\
b_2 &= 0.2^{11}/(11 \cdot 5!) \approx 1.5515 \times 10^{-11}
\end{aligned}$$

By the Alternating Series Estimation Theorem the approximation

$$\sum_{k=0}^{\infty} (-1)^k \frac{0.2^{4k+3}}{(4k+3)(2k+1)!} \approx b_0 - b_1 = \frac{0.2^3}{3} - \frac{0.2^7}{42}$$

will have an absolute error that is less than $b_2 \approx 1.5515 \times 10^{-11} < 10^{-10}$. Therefore the approximation

$$\int_0^{0.2} \sin(x^2)\, dx \approx \frac{0.2^3}{3} - \frac{0.2^7}{42} \approx 0.002666$$

has an absolute error less than 10^{-10}.

6 From $x = \sqrt[3]{t} + 4$ comes $t = (x-4)^3$. Putting this into $y = 5t - 3$ gives $y = 5(x-4)^3 - 3$. That is, the function f is given by

$$f(x) = 5(x-4)^3 - 3.$$

From $t \in [0, 27]$ we find that $x \in [4, 7]$, so the domain of f is $[4, 7]$.

7 $(2, 4\pi/3)$, $(2, -2\pi/3)$, $(-2, \pi/3)$, among other possibilities.

8 We have $r = f(\theta)$ with $f(\theta) = 8\cos\theta$. The slope m of the curve at $(4, 5\pi/6)$ is

$$\begin{aligned} m &= \frac{f'(5\pi/6)\sin(5\pi/6) + f(5\pi/6)\cos(5\pi/6)}{f'(5\pi/6)\cos(5\pi/6) - f(5\pi/6)\sin(5\pi/6)} \\ &= \frac{-8\sin(5\pi/6)\sin(5\pi/6) + 8\cos(5\pi/6)\cos(5\pi/6)}{-8\sin(5\pi/6)\cos(5\pi/6) - 8\cos(5\pi/6)\sin(5\pi/6)} \\ &= \frac{\sin^2(5\pi/6) - \cos^2(5\pi/6)}{2\cos(5\pi/6)\sin(5\pi/6)} = \frac{(1/2)^2 - (-\sqrt{3}/2)^2}{2(1/2)(-\sqrt{3}/2)} = \frac{1}{\sqrt{3}} \end{aligned}$$

9 Here $r = f(\theta)$ with $f(\theta) = 3 + 5\cos\theta$. The curve is generated for $\theta \in [0, 2\pi)$, so we find all $0 \le \theta < 2\pi$ for which

$$f'(\theta)\sin\theta + f(\theta)\cos\theta = 0,$$

which gives

$$-5\sin\theta\sin\theta + \cos\theta(3 + 5\cos\theta) = 0.$$

Since $\sin^2\theta = 1 - \cos^2\theta$, we get

$$10\cos^2\theta + 3\cos\theta - 5 = 0,$$

and so

$$\cos\theta = \frac{-3 \pm \sqrt{209}}{20} \approx -0.8728,\ 0.5728$$

by the quadratic formula. From $\cos\theta = -0.8728$ we obtain $\theta \approx 2.63, 3.65$. From $\cos\theta = 0.5728$ we obtain $\theta \approx 0.96, 5.32$. Solution set in $[0, 2\pi)$ is thus

$$\{0.96, 2.63, 3.65, 5.32\},$$

to the nearest hundredth.