Math 141 Exam #3 Key (Spring 2014)

1a We have

$$\lim_{n\to\infty} \frac{12n^5 - 4n^2}{3 - 5n - 9n^5} = \lim_{n\to\infty} \frac{12 - 4n^{-3}}{3n^{-5} - 5n^{-4} - 9} = \frac{12 - 4(0)}{3(0) - 5(0) - 9} = -\frac{12}{9} = -\frac{4}{3}.$$

1b First we evaluate

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{n} &= \lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp(\ln n^{1/n}) = \exp\left(\lim_{n\to\infty} \ln n^{1/n}\right) \\ &= \exp\left(\lim_{n\to\infty} \frac{\ln n}{n}\right) \stackrel{LR}{=} \exp\left(\lim_{n\to\infty} \frac{1}{n}\right) = \exp(0) = 1,\end{aligned}$$

where "LR" indicates an application of L'Hôpital's Rule.

Now, consider the subsequence of $\{a_n\}_{n=1}^{\infty}$ that consists of the even-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k$ for $k \geq 1$. Then, using the fact that $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{n_k} n_k^{1/n_k} = \lim_{k\to\infty} (-1)^{2k}(2k)^{1/(2k)} = \lim_{k\to\infty} (2k)^{1/(2k)} = 1.$$

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by $\{a_{n_k}\}_{k=1}^{\infty}$ with $n_k = 2k - 1$ for $k \geq 1$. Then we have

$$\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} (-1)^{2k-1}(2k-1)^{1/(2k-1)} = \lim_{k\to\infty} \left[-(2k-1)^{1/(2k-1)}\right] = -1.$$

Since $\{a_n\}$ has two subsequences with different limits, the sequence $\{a_n\}$ itself cannot converge. That is, $\{a_n\}$ diverges.

2 Starting by reindexing, we have

$$\sum_{k=1}^{\infty} 2^{-3k} = \sum_{k=0}^{\infty} 2^{-3(k+1)} = \sum_{k=0}^{\infty} 2^{-3}2^{-3k} = \sum_{k=0}^{\infty} \frac{1}{8}\left(\frac{1}{8}\right)^k = \frac{1/8}{1 - 1/8} = \frac{1}{7}.$$

3 For each $n \geq 1$ we have

$$\begin{aligned}s_n &= \sum_{k=1}^{n} \left(\frac{1}{k+5} - \frac{1}{k+6}\right) \\ &= \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) + \left(\frac{1}{n+5} - \frac{1}{n+6}\right) \\ &= \frac{1}{6} - \frac{1}{n+6},\end{aligned}$$

so

$$\sum_{k=1}^{\infty} \left(\frac{1}{k+5} - \frac{1}{k+6}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{6} - \frac{1}{n+6}\right) = \frac{1}{6}.$$

4a Since

$$\lim_{k\to\infty} \frac{k}{\sqrt{k^2+25}} = 1 \neq 0,$$

the series diverges by the Divergence Test.

4b Letting $u = -2x^2$, we have

$$\int_1^\infty xe^{-2x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-2x^2}\,dx = \lim_{b\to\infty}\int_{-2}^{-2b^2} -\frac{1}{4}e^u\,du = \lim_{b\to\infty} -\frac{1}{4}[e^u]_{-2}^{-2b^2}$$
$$= \lim_{b\to\infty} -\frac{1}{4}\left(e^{-2b^2} - e^{-2}\right) = -\frac{1}{4}(0-e^{-2}) = \frac{e^{-2}}{4},$$

so the integral

$$\int_1^\infty xe^{-2x^2}\,dx$$

converges, and therefore the series converges by the Integral Test.

4c Since

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{2^{k+1}}{(k+1)^{99}}\cdot\frac{k^{99}}{2^k}\right| = \lim_{k\to\infty} 2\left(\frac{k}{k+1}\right)^{99} = 2\left(\lim_{k\to\infty}\frac{k}{k+1}\right)^{99} = 2(1)^{99} = 2 > 1,$$

the series diverges by the Ratio Test.

4d Since

$$\lim_{k\to\infty}|a_k|^{1/k} = \lim_{k\to\infty}\left[\left(\tfrac{k}{k+1}\right)^{2k^2}\right]^{1/k} = \lim_{k\to\infty}\left(\frac{k}{k+1}\right)^{2k} = e^{-2} \approx 0.1353 < 1,$$

the series converges by the Root Test.

4e For each $k \geq 1$ we have

$$0 \leq \frac{\sin^2 k}{k\sqrt{k}} \leq \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}},$$

and since $\sum_{k=1}^\infty k^{-3/2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^\infty \frac{\sin^2 k}{k\sqrt{k}}$$

converges by the Direct Comparison Test.

4f For each $k \geq 1$ we have

$$0 \leq \frac{k^7}{k^9+3} \leq \frac{k^7}{k^9} = \frac{1}{k^2},$$

and since $\sum_{k=1}^{\infty} k^{-2}$ is a convergent p-series, it follows that

$$\sum_{k=1}^{\infty} \frac{k^7}{k^9+3}$$

converges by the Direct Comparison Test.

5a Since $\ln k$ and k are monotone increasing functions for $k \geq 2$, it follows that

$$\frac{1}{k \ln^2 k}$$

is monotone decreasing (i.e. nonincreasing) for $k \geq 2$. Also

$$\lim_{k\to\infty} \frac{1}{k \ln^2 k} = 0,$$

and so by the Alternating Series Test the series converges.

5b Since

$$\lim_{k\to\infty} \left| (-1)^k \left(1 - \frac{2}{k}\right) \right| = \lim_{k\to\infty} \left(1 - \frac{2}{k}\right) = 1 \neq 0,$$

the series diverges by the Divergence Test.