Math 141 Exam #2 Key (Spring 2014)

1a Let $u(z) = \sin^{-1} z$ and $v'(z) = 1$. Then $u'(z) = (1 - z^2)^{-1/2}$ and $v(z) = z$, and so

$$\int_{1/2}^{\sqrt{3}/2} \sin^{-1} z \, dz = \left[z \sin^{-1} z \right]_{1/2}^{\sqrt{3}/2} - \int_{1/2}^{\sqrt{3}/2} \frac{z}{\sqrt{1 - z^2}} dz.$$

Making the substitution $u = 1 - z^2$ gives

$$\begin{aligned} \int_{1/2}^{\sqrt{3}/2} \sin^{-1} z \, dz &= \frac{\sqrt{3}}{2} \sin^{-1} \frac{\sqrt{3}}{2} - \frac{1}{2} \sin^{-1} \frac{1}{2} + \frac{1}{2} \int_{3/4}^{1/4} \frac{1}{\sqrt{u}} du \\ &= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} + \frac{1}{2} \left[2\sqrt{u} \right]_{3/4}^{1/4} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{2}. \end{aligned}$$

1b Let $u(x) = \ln^2 x$ and $v'(x) = x^2$. Then $u'(x) = \frac{2}{x} \ln x$ and $v(x) = \frac{1}{3} x^3$, and so

$$\int x^2 \ln^2 x \, dx = \frac{1}{3} x^3 \ln^2 x - \frac{2}{3} \int x^2 \ln x \, dx.$$

For the integral on the right, let $u(x) = \ln x$ and $v'(x) = x^2$. Then $u'(x) = \frac{1}{x}$ and $v(x) = \frac{1}{3} x^3$, so that

$$\int x^2 \ln^2 x \, dx = \frac{1}{3} x^3 \ln^2 x - \frac{2}{3} \left(\frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^2 \, dx \right)$$

and hence

$$\int x^2 \ln^2 x \, dx = \frac{1}{3} x^3 \ln^2 x - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + c.$$

2a We have

$$\int \sin^7 x \cos^3 x \, dx = \int [1 - \cos^2 x]^3 \cos^3 x \sin x \, dx,$$

and so if we let $u = \cos x$, so that $\sin x \, dx$ is replaced by $-du$ by the Substitution Rule, we obtain

$$\begin{aligned} \int \sin^7 x \cos^3 x \, dx &= -\int (1 - u^2)^3 u^3 \, du = \int (u^3 - 3u^5 + 3u^7 - u^9) \, du \\ &= \frac{1}{4} u^4 - \frac{1}{2} u^6 + \frac{3}{8} u^8 - \frac{1}{10} u^{10} + c \\ &= \frac{1}{4} \cos^4 x - \frac{1}{2} \cos^6 x + \frac{3}{8} \cos^8 x - \frac{1}{10} \cos^{10} x + c. \end{aligned}$$

Alternatively: write

$$\int \sin^7 x [1 - \sin^2 x] \cos x \, dx,$$

and let $u = \sin x$ to obtain

$$\int u^7 (1 - u^2) \, du = \frac{1}{8} u^8 - \frac{1}{10} u^{10} + c = \frac{1}{8} \sin^8 x - \frac{1}{10} \sin^{10} x + c.$$

2b We have

$$\int \frac{\csc^4 t}{\cot^2 t} dt = \int \csc^2 t \cdot \frac{\cot^2 t + 1}{\cot^2 t} dt.$$

Substitute $u = \cot t$, so that formally $\csc^2 t\, dt = -du$, and we obtain

$$\int \csc^2 t \cdot \frac{\cot^2 t + 1}{\cot^2 t} dt = -\int \frac{u^2+1}{u^2} du = -\int (1 + u^{-2}) du = -u + \frac{1}{u} + c$$
$$= -\cot t + \tan t + c.$$

2c With a basic trigonometric identity we get

$$\int_0^{\pi/2} \sqrt{1 - \cos 2x}\, dx = \int_0^{\pi/2} \sqrt{2\sin^2 x}\, dx = \sqrt{2} \int_0^{\pi/2} \sin x\, dx = \sqrt{2}.$$

3a Let $x = \frac{1}{3}\tan\theta$. Formally we obtain $dx = \frac{1}{3}\sec^2\theta d\theta$, and also $\sqrt{9x^2+1} = \sec\theta$. Running through the usual trigonometric substitution process yields

$$\int_0^{1/3} \frac{1}{(9x^2+1)^{3/2}} dx = \frac{\sqrt{2}}{6}.$$

3b Let $t = 13\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so that dt is replaced with $13\cos\theta\, d\theta$ as part of the substitution. Observe that $-\pi/2 \le \theta \le \pi/2$ implies $\cos\theta \ge 0$, so that

$$\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta.$$

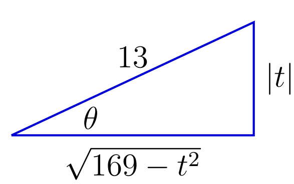
Now,

$$\int \sqrt{169 - t^2}\, dt = \int \sqrt{169 - 169\sin^2\theta} \cdot 13\cos\theta\, d\theta = \int 169\cos\theta\sqrt{1 - \sin^2\theta}\, d\theta$$
$$= 169 \int \cos\theta\sqrt{\cos^2\theta}\, d\theta = 169 \int \cos^2\theta\, d\theta,$$

and with the deft use of the given formula for $\int \cos^n\theta\, d\theta$ we obtain

$$\int \sqrt{169 - t^2}\, dt = 169\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int d\theta\right) = \frac{169}{2}\cos\theta\sin\theta + \frac{169}{2}\theta + c.$$

From $t = 13\sin\theta$ comes $\sin\theta = t/13$, so $\theta = \sin^{-1}(t/13)$ and θ may be characterized as an angle in the right triangle



Note that $t \geq 0$ if $\theta \in [0, \pi/2]$, and $t < 0$ if $\theta \in [-\pi/2, 0)$. From this triangle we see that $\cos\theta = \sqrt{169 - t^2}/13$, and therefore

$$\begin{aligned}\int \sqrt{169 - t^2}\, dt &= \frac{169}{2} \cdot \frac{\sqrt{169 - t^2}}{13} \cdot \frac{t}{13} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c \\ &= \frac{t\sqrt{169 - t^2}}{2} + \frac{169}{2} \sin^{-1}\left(\frac{t}{13}\right) + c.\end{aligned}$$

4a We have

$$\frac{2}{x^3 + x^2} = \frac{2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},$$

whence we obtain

$$2 = Ax(x+1) + B(x+1) + Cx^2 = (A + C)x^2 + (A + B)x + B,$$

which implies we must have $A + C = 0$, $A + B = 0$, and $B = 2$. The only solution is $(A, B, C) = (-2, 2, 2)$. Hence

$$\begin{aligned}\int \frac{2}{x^3 + x^2} dx = \int \left(-\frac{2}{x} + \frac{2}{x^2} + \frac{2}{x+1}\right) dx &= -2\ln|x| - \frac{2}{x} + 2\ln|x+1| + c \\ &= \ln\left(\frac{x+1}{x}\right)^2 - \frac{2}{x} + c.\end{aligned}$$

4b Again start with a decomposition, noting that $x^2 + 2x + 6$ is an irreducible quadratic:

$$\begin{aligned}\int \frac{2}{(x-4)(x^2 + 2x + 6)}\, dx &= \int \left(\frac{1/15}{x-4} + \frac{-x/15 - 2/5}{x^2 + 2x + 6}\right) dx \\ &= \frac{1}{15} \ln|x - 4| - \frac{1}{15} \int \frac{x+6}{(x+1)^2 + 5}\, dx. \qquad (1)\end{aligned}$$

For the remaining integral, let $u = x + 1$ to obtain

$$\int \frac{x+6}{(x+1)^2 + 5}\, dx = \int \frac{u+5}{u^2+5}\, du = \int \frac{u}{u^2+5}\, du + 5 \int \frac{1}{u^2 + (\sqrt{5}\,)^2}\, du \qquad (2)$$

Letting $w = u^2 + 5$ in the first integral at right in (2), and using Formula (9) for the second, we next get

$$\begin{aligned}\int \frac{x+6}{(x+1)^2 + 5}\, dx &= \int \frac{1/2}{w}\, dw + 5 \cdot \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2} \ln|w| + \sqrt{5} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c = \frac{1}{2} \ln(u^2 + 5) + \sqrt{5} \tan^{-1}\left(\frac{u}{\sqrt{5}}\right) + c \\ &= \frac{1}{2} \ln[(x+1)^2 + 5] + \sqrt{5} \tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\end{aligned}$$

Returning to (1),

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx = \frac{\ln|x-4|}{15} - \frac{1}{15}\left[\frac{\ln[(x+1)^2+5]}{2} + \sqrt{5}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right]$$
$$= \frac{\ln|x-4|}{15} - \frac{\ln(x^2+2x+6)}{30} - \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c.$$

5 Make the substitution $u = \pi/x$, so that

$$\int_2^\infty \frac{\cos(\pi/x)}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{\cos(\pi/x)}{x^2}dx = \lim_{b\to\infty}\int_{\pi/2}^{\pi/b} -\frac{\cos u}{\pi}du = \lim_{b\to\infty}\frac{1-\sin(\pi/b)}{\pi} = \frac{1}{\pi},$$

which shows that the integral converges.

6 The volume of the solid is

$$\mathcal{V} = \int_1^\infty \pi[f(x)]^2dx = \pi\int_1^\infty \frac{x+1}{x^3}dx = \pi\lim_{b\to\infty}\int_2^b \frac{1}{x^4}dx = \pi\lim_{b\to\infty}\left[-\tfrac{1}{3}x^{-3}\right]_2^b = \pi\lim_{b\to\infty}\tfrac{1}{3}(2^{-3}-b^{-3})$$
$$= \pi\cdot\tfrac{1}{3}(2^{-3}-0) = \frac{\pi}{24}.$$

7 The integral is improper since $\ln(0)$ is undefined. Using integration by parts with $u(x) = \ln x$ and $v'(x) = x$ gives

$$\int_0^1 x\ln x\,dx = \lim_{a\to0^+}\int_a^1 x\ln x\,dx = \lim_{a\to0^+}\left(\frac{1}{2}\left[x^2\ln x\right]_a^1 - \int_a^1 \frac{x}{2}dx\right)$$
$$= \lim_{a\to0^+}\left[-\frac{1}{2}a^2\ln a - \frac{1}{4}(1-a^2)\right] = -\frac{1}{4},$$

where by L'Hôpital's Rule we have

$$\lim_{a\to0^+} a^2\ln a = \lim_{a\to0^+}\frac{\ln a}{a^{-2}} = \lim_{a\to0^+}\frac{a^{-1}}{-2a^{-3}} = \lim_{a\to0^+}\left(-\frac{a^2}{2}\right) = 0.$$