Math 141 Exam #1 Key (Spring 2014)

1 We have

$$f(2) = 2(2)^3 + 2 - 12 = 18 - 12 = 6,$$

and from $f'(x) = 6x^2 + 1$ we find that $f'(2) = 25 \neq 0$. Now, clearly f is differentiable on $(-\infty, \infty)$, and since $f' > 0$ on $(-\infty, \infty)$ we conclude that f is everywhere increasing and therefore one-to-one. By the appropriate theorem we then obtain

$$(f^{-1})'(6) = \frac{1}{f'(2)} = \frac{1}{25}.$$

2 Let f_1 be the restriction of f to the interval $[6, \infty)$. That is, $f_1(x) = f(x)$ for $x \geq 6$. Then f_1 is a one-to-one function and thus has an inverse f_1^{-1}. To find f_1^{-1} set $y = f_1(x)$, so that $y = (x-6)^2$ for $x \geq 6$. Then

$$\sqrt{y} = |x - 6| = x - 6,$$

whence $x = 6 + \sqrt{y}$. Since $y = f_1(x)$ if and only if $x = f_1^{-1}(y)$, we obtain $f_1^{-1}(y) = 6 + \sqrt{y}$.

Next, let f_2 be the restriction of f to the interval $(-\infty, 6]$. That is, $f_2(x) = f(x)$ for $x \leq 6$. Then f_2 is a one-to-one function and has an inverse f_2^{-1}. To find f_2^{-1} set $y = f_2(x)$, so that $y = (x-6)^2$ for $x \leq 6$. Then

$$\sqrt{y} = |x - 6| = -(x - 6) = 6 - x,$$

whence $x = 6 - \sqrt{y}$. Since $y = f_2(x)$ if and only if $x = f_2^{-1}(y)$, we obtain $f_2^{-1}(y) = 6 - \sqrt{y}$.

We have now found that there are two (local) inverses associated with f: the function f_1^{-1} given by

$$f_1^{-1}(y) = 6 + \sqrt{y}$$

with $\text{Dom}(f_1^{-1}) = \text{Ran}(f_1) = [0, \infty)$, and f_2^{-1} given by

$$f_2^{-1}(y) = 6 - \sqrt{y}$$

with $\text{Dom}(f_2^{-1}) = \text{Ran}(f_2) = [0, \infty)$.

3a $f'(x) = \dfrac{2e^{2x}}{e^{2x} + 3}$

3b $\text{Dom}(g) = (0, \infty)$, and for all $x > 0$ we have

$$g(x) = x^{\ln(x^5)} = \exp\left(\ln\left(x^{\ln(x^5)}\right)\right) = \exp\big(\ln(x^5)\ln(x)\big) = \exp\big(5\ln^2(x)\big),$$

and thus

$$g'(x) = \exp\big(5\ln^2(x)\big) \cdot \big(5\ln^2(x)\big)' = x^{\ln(x^5)} \cdot \frac{10\ln(x)}{x} = \frac{10x^{\ln(x^5)}\ln(x)}{x}.$$

3c For x such that $\sin x > 0$ we have

$$h(x) = (\sin x)^{\tan x} = \exp\big(\ln((\sin x)^{\tan x})\big) = \exp(\tan x \cdot \ln(\sin x)),$$

and thus

$$\begin{aligned}
h'(x) &= \exp(\tan x \cdot \ln(\sin x)) \cdot (\tan x \cdot \ln(\sin x))' \\
&= \exp(\tan x \cdot \ln(\sin x)) \cdot \Big(\tan x \cdot \frac{\cos x}{\sin x} + \sec^2 x \cdot \ln(\sin x)\Big) \\
&= (\sin x)^{\tan x}\Big(1 + \ln(\sin x)^{\sec^2 x}\Big)
\end{aligned}$$

3d $k'(x) = \dfrac{7}{(4-x^5)\ln(3)} \cdot (4-x^5)' = -\dfrac{35x^4}{(4-x^5)\ln(3)}$

3e $\ell'(x) = \dfrac{1}{e^{-2x}\sqrt{(e^{-2x})^2-1}} \cdot (e^{-2x})' = \dfrac{-2e^{-2x}}{e^{-2x}\sqrt{e^{-4x}-1}} = -\dfrac{2}{\sqrt{e^{-4x}-1}}$

3f $p'(x) = -\dfrac{1}{1+(\sqrt{x}\,)^2} \cdot (\sqrt{x}\,)' = -\dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2\sqrt{x}(1+x)}$

4a $\displaystyle\int (3e^{-8x} - 8e^{11x})dx = -\frac{3}{8}e^{-8x} - \frac{8}{11}e^{11x} + c$

4b $\displaystyle\int \frac{9}{4-9y}dy = -\ln|4-9y| + c$

4c Let $u = x^4$, so by the Substitution Rule we replace $x^3\,dx$ with $\frac{1}{4}du$ to get

$$\int x^3 10^{x^4}\,dx = \frac{1}{4}\int 10^u\,du = \frac{1}{4} \cdot \frac{10^u}{\ln 10} + c = \frac{10^{x^4}}{4\ln 10} + c.$$

5a Let $u = \ln(x)$, so when $x = 1$ we have $u = \ln(1) = 0$, and when $x = 3e$ we have $u = \ln(3e)$. Now, by the Substitution Rule we replace $\frac{1}{x}dx$ with du to get

$$\int_0^{\ln(3e)} \frac{e^u}{2}\,du = \left[\frac{1}{2}e^u\right]_0^{\ln(3e)} = \frac{1}{2}(e^{\ln(3e)} - e^0) = \frac{3e-1}{2}.$$

5b We have

$$5\int_2^{2\sqrt{3}} \frac{1}{z^2+2^2}dz = 5\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_2^{2\sqrt{3}} = \frac{5}{2}\Big[\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)\Big] = \frac{5}{2}\Big(\frac{\pi}{3} - \frac{\pi}{4}\Big) = \frac{5\pi}{24}.$$

6 For all $x > 0$ we have

$$\left(\frac{2}{3x}\right)^{8/x} = \exp\left[\ln\left(\frac{2}{3x}\right)^{8/x}\right] = \exp\left[\frac{8}{x}\ln\left(\frac{2}{3x}\right)\right] = \exp\left(\frac{8\ln(2/3x)}{x}\right).$$

The functions $f(x) = 8\ln(2/3x)$ and $g(x) = x$ are differentiable on $(0, \infty)$, and $g'(x) = 1 \neq 0$ for all $x \in (0, \infty)$. Since $g(x) \to \infty$ as $x \to \infty$, and

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-8/x}{1} = 0,$$

by L'Hôpital's Rule we obtain

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{8\ln(2/3x)}{x} = 0$$

as well. Now, since $\exp(x)$ is a continuous function,

$$\lim_{x\to\infty}\left(\frac{2}{3x}\right)^{8/x} = \lim_{x\to\infty} \exp\left(\frac{8\ln(2/3x)}{x}\right) = \exp\left(\lim_{x\to\infty} \frac{8\ln(2/3x)}{x}\right) = \exp(0) = 1.$$

7 We find that $x^{50}, 1.0005^x \to \infty$ as $x \to \infty$. To determine which grows faster, x^{50} or 1.0005^x, we must evaluate

$$\lim_{x\to\infty} \frac{1.0005^x}{x^{50}}.$$

If the limit equals 0 then x^{50} grows faster, and if the limit equals ∞, then 1.0005^x grows faster. We have

$$\begin{aligned}\lim_{x\to\infty} \ln\left(\frac{1.0005^x}{x^{50}}\right) &= \lim_{x\to\infty}\left(\ln 1.0005^x - \ln x^{50}\right) = \lim_{x\to\infty}(x\ln 1.0005 - 50\ln x) \\ &= \lim_{x\to\infty} x\left(\ln 1.0005 - \frac{50\ln x}{x}\right),\end{aligned}$$

where we easily find by L'Hôpital's Rule that

$$\lim_{x\to\infty} \frac{50\ln x}{x} = \lim_{x\to\infty} \frac{50/x}{1} = 0,$$

so that

$$\lim_{x\to\infty}\left(\ln 1.0005 - \frac{50\ln x}{x}\right) = \ln 1.0005 > 0$$

and therefore

$$\lim_{x\to\infty} \ln\left(\frac{1.0005^x}{x^{50}}\right) = \lim_{x\to\infty} x\left(\ln 1.0005 - \frac{50\ln x}{x}\right) = (\infty)(\ln 1.0005) = \infty.$$

Now,

$$\lim_{x\to\infty} \frac{1.0005^x}{x^{50}} = \lim_{x\to\infty} \exp\left[\ln\left(\frac{1.0005^x}{x^{50}}\right)\right] = \exp\left[\lim_{x\to\infty} \ln\left(\frac{1.0005^x}{x^{50}}\right)\right] = \exp(\infty) = \infty,$$

so 1.0005^x grows faster than x^{50} and we write $1.0005^x \gg x^{50}$.