Math 141 Exam #4 Key (Spring 2011)

1. The Root Test will do nicely here: $\lim_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\left|\frac{(x-1)^k k^k}{(k+1)^k}\right|^{1/k} = \lim_{k\to\infty}\frac{k}{k+1}|x-1| = |x-1|$, so the series will converge if $|x-1|<1$ (and diverge if $|x-1|>1$), implying the radius of convergence is $R=1$. That is, the series converges for all $x\in(0,2)$, and it remains to investigate the endpoints. When $x=2$ the series becomes $\sum_{k=0}^{\infty}\frac{k^k}{(k+1)^k}$, but since $\lim_{k\to\infty}\frac{k^k}{(k+1)^k}=1/e\neq 0$, the series diverges by the Divergence Test. When $x=0$ the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^k k^k}{(k+1)^k}$, but again $\lim_{k\to\infty}\frac{(-1)^k k^k}{(k+1)^k}\neq 0$ so the series diverges. Therefore the interval of convergence is $(0,2)$.

2. We use $\frac{1}{x+1}=\sum_{k=0}^{\infty}(-1)^k x^k$ to get $\frac{1}{1+x^2}=\sum_{k=0}^{\infty}(-1)^k(x^2)^k=\sum_{k=0}^{\infty}(-1)^k x^{2k}$. Now, $\lim_{k\to\infty}\sqrt[k]{|a_k|}=\lim_{k\to\infty}\left|x^{2k}\right|^{1/k}=\lim_{k\to\infty}x^2=x^2$, so by the Root Test the series converges if $x^2<1$, which gives $-1<x<1$. If $x=\pm1$ the series becomes $\sum_{k=0}^{\infty}(-1)^k$, which diverges by the Divergence Test. Hence the interval of convergence is $(-1,1)$.

3a. From $f(x)=e^{-3x}$, $f'(x)=-3e^{-3x}$, $f''(x)=9e^{-3x}$, $f'''(x)=-27e^{-3x}$ we get

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots=1-3x+\frac{9}{2}x^2-\frac{9}{2}x^3+\cdots$$

3b. Power series for e^{-3x} is $\sum_{k=0}^{\infty}\frac{(-1)^k 3^k}{k!}x^k$

3c. Use Ratio Test: $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\left|\frac{(-1)^{k+1}3^{k+1}x^{k+1}}{(k+1)!}\cdot\frac{k!}{(-1)^k 3^k x^k}\right|=\lim_{k\to\infty}\left|\frac{3x}{k+1}\right|=0$ for all $x\in\mathbb{R}$, so the interval of convergence is $(-\infty,\infty)$.

4. Use $\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ to get $\sin x^2=\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!}$, which converges for all $x\in\mathbb{R}$. Hence integration can be done "termwise" in the natural way: $\int_0^{0.2}\sin x^2\,dx=\int_0^{0.2}\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!}dx=\sum_{k=0}^{\infty}\left[\int_0^{0.2}\frac{(-1)^k x^{4k+2}}{(2k+1)!}dx\right]$ $=\sum_{k=0}^{\infty}\frac{(-1)^k x^{4k+3}}{(4k+3)(2k+1)!}\Big|_0^{0.2}=\frac{1}{375}-3.048\times10^{-7}+\cdots$. The series is a convergent alternating series, so the error in estimating its value by taking the first n terms will be less than the value of the $(n+1)$st term. So here, to estimate the value of the series with an error less than 10^{-4}, we only need the first term: $1/375$.

5. From $y=t+2$ we get $t=y-2$, and then $x=(t+1)^2$ becomes $x=(y-1)^2$.

6. $(-3,-\pi/3)$ and $(3,-4\pi/3)$.

7. The first thing to notice is that any point where $\theta = \pi/2$ will satisfy the equation, which corresponds to the vertical line $x = 0$. Assuming we're not on this line, we have $x \neq 0$ and thus $r \neq 0$, which then implies $\cos\theta = x/r$ and $\sin\theta = y/r$, and so (recalling $r^2 = x^2 + y^2$ and $\sin 2\theta = 2\sin\theta\cos\theta$), we find that $r\cos\theta = \sin(2\theta)$ $\Rightarrow\ x = \dfrac{2xy}{r^2}\ \Rightarrow\ x = \dfrac{2xy}{x^2+y^2}\ \Rightarrow\ 1 = \dfrac{2y}{x^2+y^2}\ \Rightarrow\ x^2 + (y-1)^2 = 1$. This is a circle centered at $(0,1)$ with radius 1. So, the graph of $r\cos\theta = \sin(2\theta)$ is as pictured.