Math 141 Exam #3 Key (Spring 2011)

1. 243 and 729. Recurrence relation: $a_{n+1} = 3a_n$, $a_1 = 1$. Explicit formula: $a_n = 3^{n-1}$ for $n \geq 1$.

2. We obtain $\lim_{n\to\infty}(1+5/n)^n = \lim_{n\to\infty}\exp\left[\ln(1+5/n)^n\right] = \exp\left[\lim_{n\to\infty}\ln(1+5/n)^n\right] = \exp\left[\lim_{n\to\infty}\frac{\ln(1+5/n)}{1/n}\right]$ $= \exp\left[\lim_{n\to\infty}\frac{(1+5/n)^{-1}\cdot(-5/n^2)}{-1/n^2}\right] = \exp\left[\lim_{n\to\infty}\frac{5}{1+5/n}\right] = \exp(5) = e^5$, using L'Hôpital's Rule en route.

3. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{4^k} = \sum_{k=0}^{\infty}\frac{1}{4^{k+2}} = \sum_{k=0}^{\infty}\frac{1}{4^2}\cdot\left(\frac{1}{4}\right)^k = \frac{1/16}{1-1/4} = \frac{1}{12}.$

4. $\displaystyle s_n = \sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right) = \left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$ $\displaystyle = \frac{1}{3}-\frac{1}{n+3}$, so $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k+2}-\frac{1}{k+3}\right) = \lim_{n\to\infty}s_n = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{n+3}\right) = \frac{1}{3}.$

5. Here $\displaystyle\lim_{k\to\infty}a_k = \lim_{k\to\infty}\frac{k^3}{k^3+10} = 1 \neq 0$, and so the series diverges by the Divergence Test.

6. Let $f(x) = xe^{-3x^2}$, which clearly is continuous and nonnegative on $[1,\infty)$. Now, $f'(x) = (1-6x^2)e^{-3x^2}$, so we have $f'(x) < 0$ for $x > 1/\sqrt{6}$ or $x < -1/\sqrt{6}$, which certainly shows that f is nonincreasing on $[1,\infty)$ as well. The hypotheses of the Integral Test are therefore satisfied for $N = 1$.

Making the substitution $u = -3x^2$, we obtain

$$\int_1^{\infty} xe^{-3x^2}dx = \lim_{b\to\infty}\int_1^b xe^{-3x^2}dx = \lim_{b\to\infty}\int_{-3}^{-3b^2} -\frac{1}{6}e^u du = \lim_{b\to\infty}\left[-\frac{1}{6}\left(e^{-3b^2}-e^{-3}\right)\right] = \frac{1}{6e^3},$$

and thus the integral $\int_1^{\infty} xe^{-3x^2}dx$ converges. Therefore by the Integral Test $\sum_{k=1}^{\infty} ke^{-3k^2}$ converges as well.

7. Here $\displaystyle a_k = \frac{k^6}{k!}$, so $\displaystyle r = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{(k+1)^6}{(k+1)!}\cdot\frac{k!}{k^6} = \lim_{k\to\infty}\frac{(k+1)^5}{k^6} = 0 < 1$ and by the Ratio Test the series converges.

8. $\displaystyle r = \lim_{k\to\infty}\sqrt[k]{a_k} = \lim_{k\to\infty}\sqrt[k]{\left(\frac{k+1}{2k}\right)^k} = \lim_{k\to\infty}\frac{k+1}{2k} = \frac{1}{2} < 1$, and so by the Root Test the series converges.

9. For all $k \geq 1$ we have $|\sin(1/k)| \leq 1$, and thus $\displaystyle\left|\frac{\sin(1/k)}{k^2}\right| \leq \frac{1}{k^2}$. Now, since the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges (it's a p-series with $p = 2$), by the Comparison Test the series $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(1/k)}{k^2}$ converges.

10. It diverges by the Divergence Test, since the limit $\displaystyle\lim_{k\to\infty}(-1)^k\frac{k^2-1}{k^2+3}$ does not exist (and therefore does not equal 0).