Math 141 Exam #2 Key (Spring 2011)

1. $\displaystyle\int \sec^5 x\,dx = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\left(\frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\,dx\right) = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\left(\frac{\sec x\tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|\right) + C = \frac{\sec^3 x\tan x}{4} + \frac{3\sec x\tan x}{8} + \frac{3}{8}\ln|\sec x + \tan x| + C.$

2. $\displaystyle\int \sin^3 x\cos^5 x\,dx = \int \sin^3 x(1-\sin^2 x)^2\cos x\,dx$, so if we let $u = \sin x$ then $du = \cos x\,dx$ and the integral becomes $\displaystyle\int u^3(1-u^2)^2\,du = \int (u^3 - 2u^5 + u^7)\,du = \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C = \frac{1}{4}\sin^4 x - \frac{1}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C.$ Alternatively, letting $u = \cos x$: $\displaystyle\int \sin^3 x\cos^5 x\,dx = \int \sin^2 x\cos^5 x\cdot\sin x\,dx = \int (\cos^5 x - \cos^7 x)\sin x\,dx = \int (u^7 - u^5)du = \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.$

3. Let $x = 2\tan\theta$ for $\theta \in (-\pi/2, \pi/2)$, so $dx = 2\sec^2\theta\,d\theta$ and $\displaystyle\int \frac{1}{\sqrt{16+4x^2}}\,dx = \int \frac{2\sec^2\theta}{\sqrt{16+16\tan^2\theta}}\,d\theta = \frac{1}{2}\int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C.$

4. Let $x = \frac{3}{2}\sin\theta$ for $\theta \in [-\pi/2, \pi/2]$, so $dx = \frac{3}{2}\cos\theta\,d\theta$ and $\displaystyle\int \sqrt{9-4x^2}\,dx = \frac{3}{2}\int \cos\theta\sqrt{9-9\sin^2\theta}\,d\theta = \frac{9}{2}\int \cos\theta\sqrt{1-\sin^2\theta}\,d\theta = \frac{9}{2}\int \cos\theta\sqrt{\cos^2\theta}\,d\theta = \frac{9}{2}\int \cos^2\theta\,d\theta = \frac{9}{4}\int (1+\cos 2\theta)\,d\theta = \frac{9}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{9}{4}(\theta + \sin\theta\cos\theta) + C = \frac{9}{4}\left[\sin^{-1}\left(\frac{2x}{3}\right) + \frac{2x}{3}\cdot\frac{\sqrt{9-4x^2}}{3}\right] + C = \frac{9}{4}\sin^{-1}\left(\frac{2x}{3}\right) + \frac{x}{2}\sqrt{9-4x^2} + C.$

5. Decompose: $\displaystyle\frac{1}{t^2-9} = \frac{A}{t-3} + \frac{B}{t+3}$, which gives $(A+B)t + (3A-3B) = 1$ and thus $A+B = 0$ and $3A-3B = 1$. Solving this system of equations yields $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. Now, $\displaystyle\int \frac{1}{t^2-9}\,dt = \frac{1}{6}\int\left(\frac{1}{t-3} - \frac{1}{t+3}\right)dt = \frac{1}{6}\left[\ln|t-3| - \ln|t+3|\right] + C.$

6. Decompose: $\displaystyle\frac{y}{(y-6)(y+2)^2} = \frac{A}{y-6} + \frac{B}{y+2} + \frac{C}{(y+2)^2} \Rightarrow y = A(y+2)^2 + B(y-6)(y+2) + C(y-6) \Rightarrow (A+B)y^2 + (4A-4B+C)y + (4A-12B-6C) = y.$ We obtain the system of equations

$$\begin{cases} A+B=0 \\ 4A-4B+C=1 \\ 4A-12B-6C=0 \end{cases}$$

Solving this gives $A = \frac{3}{32}$, $B = -\frac{3}{32}$, $C = \frac{1}{4}$, and so $\displaystyle\int \frac{y}{(y-6)(y+2)^2}\,dy = \frac{3}{32}\int \frac{1}{y-6}\,dy - \frac{3}{32}\int \frac{1}{y+2}\,dy +$

$$\frac{1}{4}\int \frac{1}{(y+2)^2}\,dy = \frac{3}{32}\ln|y-6| - \frac{3}{32}\ln|y+2| - \frac{1}{4(y+2)} + K.$$

7. Decompose: $\dfrac{z+1}{z(z^2+4)} = \dfrac{A}{z} + \dfrac{Bz+C}{z^2+4} \;\Rightarrow\; z+1 = A(z^2+4) + (Bz+C)z \;\Rightarrow\; (A+B)z^2 + Cz + 4A = z+1$, and so we must have $A+B=0$, $C=1$, and $4A=1$. Solving the system yields $A = \dfrac{1}{4}$, $B = -\dfrac{1}{4}$, $C=1$. Hence $\displaystyle\int \frac{z+1}{z(z^2+4)}\,dz = \frac{1}{4}\int \frac{1}{z}\,dz - \frac{1}{4}\int \frac{z}{z^2+4}\,dz + \int \frac{1}{z^2+4}\,dz = \frac{1}{4}\ln|z| - \frac{1}{8}\ln(z^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right) + K.$

8. Let $u = \dfrac{\pi}{x}$, so $du = -\dfrac{\pi}{x^2}\,dx \;\Rightarrow\; -\dfrac{1}{3\pi}\,du = \dfrac{1}{3x^2}\,dx$ and we get $\displaystyle\int_2^{\infty} \frac{\cos(\pi/x)}{3x^2}\,dx = \lim_{b\to\infty}\int_2^{b} \frac{\cos(\pi/x)}{3x^2}\,dx = \lim_{b\to\infty}\int_{\pi/2}^{\pi/b} -\frac{\cos u}{3\pi}\,du = \lim_{b\to\infty} \frac{-1}{3\pi}\left[\sin u\right]_{\pi/2}^{\pi/b} = \lim_{b\to\infty}\left[-\frac{1}{3\pi}\left(\sin\frac{\pi}{b} - 1\right)\right] = -\frac{1}{3\pi}(\sin 0 - 1) = \frac{1}{3\pi}.$