Math 141 Exam #1 Key (Spring 2011)

1. The relationship is $f(x) = y \Leftrightarrow f^{-1}(y) = x$, so suppose $f(x) = y$. Then $y = \sqrt{x+3} \Rightarrow y^2 = x+3 \Rightarrow x = y^2 - 3$, and since $f^{-1}(y) = x$ we obtain $f^{-1}(y) = y^2 - 3$. (Note: the condition $x \geq 3$ implies that $y \geq 0$, which is to say we have $\text{Dom}(f^{-1}) = [0, \infty)$ and *not* all reals.) Replacing y with x gives $f^{-1}(x) = x^2 - 3$.

2. We're not able to get f^{-1} directly, so we must employ the theorem as follows: "If f is one-to-one and differentiable on an open interval I, $a \in I$, and $f(a) = b$, then $(f^{-1})'(b) = 1/f'(a)$ if $f'(a) \neq 0$." First we must find a for which $f(a) = 3$, which requires solving $a^3 + a + 1 = 3$. This is a knotty equation to solve analytically but one obvious solution is $a = 1$, and actually this is the *only* real solution since f (which is differentiable everywhere) is seen to be one-to-one by examining its derivative: $f'(x) = 3x^2 + 1 > 0$ for all $x \in \mathbb{R}$, so f must be strictly increasing on $\mathbb{R}$. Now, since $f(1) = 3$, we have $(f^{-1})'(3) = 1/f'(1) = 1/4$.

3. If u is a differentiable function of x, then $\dfrac{d}{dx}(\ln|u|) = \dfrac{1}{u}\dfrac{du}{dx}$ wherever $u(x) \neq 0$. (This formula is important because it provides the basis for logarithmic differentiation.) So $\dfrac{d}{dx}(\ln|x^2-1|) = \dfrac{1}{x^2-1}\dfrac{d}{dx}(x^2-1) = \dfrac{2x}{x^2-1}$ wherever $x^2 \neq 1$, so the result is valid on $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$.

4. $f'(x) = (2\cos 2x)e^{\sin 2x}$, so $f'(\pi/4) = \left(2\cos\frac{\pi}{2}\right)e^{\sin(\pi/2)} = 0$.

5a. Substitution: let $u = \sqrt{x}$, so $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$ gives $2\,du = \dfrac{1}{\sqrt{x}}dx$, and therefore $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int 2e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$.

5b. $\displaystyle\int_{-1}^{1} 10^x dx = \left[\frac{1}{\ln 10}10^x\right]_{-1}^{1} = \frac{1}{\ln 10}\left(10 - 10^{-1}\right) = \frac{9.9}{\ln 10}$.

5c. $\displaystyle\int \frac{5}{\sqrt{7^2 - x^2}}\,dx = 5\sin^{-1}\left(\frac{x}{7}\right) + C$.

6. $\dfrac{d}{dx}(\ln f(x)) = \dfrac{d}{dx}\left(\ln(\cos x)^{\tan x}\right) \Rightarrow \dfrac{f'(x)}{f(x)} = \dfrac{d}{dx}(\tan x \cdot \ln(\cos x)) \Rightarrow \dfrac{f'(x)}{f(x)} = \ln(\cos x)\cdot\sec^2 x - \tan^2 x \Rightarrow f'(x) = (\cos x)^{\tan x}\left(\sec^2 x \cdot \ln(\cos x) - \tan^2 x\right)$. Keep in mind that this result is valid only for $x \in \text{Dom}(f)$ where $f(x) \neq 0$.

7a. $y' = 4^{-x}\cos x - (\ln 4)4^{-x}\sin x$.

7b. Note $y = \pi\ln(x^3+1)$, so $y' = \dfrac{\pi}{x^3+1}\cdot(3x^2) = \dfrac{3\pi x^2}{x^3+1}$.

7c. $y' = \dfrac{4}{(x^2-1)\ln 3}\cdot\dfrac{d}{dx}(x^2-1) = \dfrac{8x}{(x^2-1)\ln 3}$.

7d. $f'(z) = \dfrac{1}{1+(2z^2-4)^2} \cdot \dfrac{d}{dz}(2z^2-4) = \dfrac{4z}{4z^4-16z^2+17}$.

8. $\lim_{x\to 0^+} x^{20x} = \lim_{x\to 0^+} \exp(\ln x^{20x}) = \exp\left(\lim_{x\to 0^+} 20x \ln x\right) = \exp\left(\lim_{x\to 0^+} \dfrac{20\ln x}{1/x}\right) = \exp\left(\lim_{x\to 0^+} \dfrac{(20\ln x)'}{(1/x)'}\right) = \exp\left(\lim_{x\to 0^+} \dfrac{20/x}{-1/x^2}\right) = \exp\left(\lim_{x\to 0^+} \dfrac{20x}{-1}\right) = \exp(0) = 1$, by way of the arguments exhibited in the notes.

9a. Apply integration by parts twice. For the first round let $u(x) = x^2$ and $v'(x) = e^{4x}$, so $u'(x) = 2x$ and $v(x) = \frac{1}{4}e^{4x}$, and we obtain $\int x^2 e^{4x} dx = \dfrac{x^2}{4}e^{4x} - \int \dfrac{2x}{4} e^{4x} dx = \dfrac{x^2}{4}e^{4x} - \dfrac{1}{2}\int x e^{4x} dx$. For the second round let $u(x) = x$ and $v'(x) = e^{4x}$, so $u'(x) = 1$ and $v(x) = \frac{1}{4}e^{4x}$, and we get $\int x e^{4x} dx = \dfrac{x}{4}e^{4x} - \int \dfrac{1}{4} e^{4x} dx = \dfrac{x}{4}e^{4x} - \dfrac{1}{16}e^{4x}$. Putting this into our previous result yields $\int x^2 e^{4x} dx = \dfrac{x^2}{4}e^{4x} - \int \dfrac{2x}{4} e^{4x} dx = \dfrac{x^2}{4}e^{4x} - \dfrac{1}{2}\left(\dfrac{x}{4}e^{4x} - \dfrac{1}{16}e^{4x}\right) + C = \dfrac{x^2}{4}e^{4x} - \dfrac{x}{8}e^{4x} + \dfrac{1}{32}e^{4x} + C = \left(\dfrac{x^2}{4} - \dfrac{x}{8} + \dfrac{1}{32}\right)e^{4x} + C$.

9b. Let $u(x) = x$, $v'(x) = \cos 2x$, so $u'(x) = 1$, $v(x) = \frac{1}{2}\sin 2x$ and integration by parts yields $\int_0^{\pi/2} x \cos 2x \, dx = \left[\dfrac{x}{2}\sin 2x\right]_0^{\pi/2} - \dfrac{1}{2}\int_0^{\pi/2} \sin 2x \, dx = 0 - \dfrac{1}{2}\left[-\dfrac{1}{2}\cos 2x\right]_0^{\pi/2} = \dfrac{1}{4}(\cos\pi - \cos 0) = -\dfrac{1}{2}$.