Math 141 Exam #2 Key (Fall 2021)

1 Let $u = \sqrt{x}$, so $u^3 = x^{3/2}$ and $dx = 2u\,du$:

$$\int \frac{2u}{u+u^3}du = \int \frac{2}{1+u^2}du = 2\arctan u + c = 2\arctan\sqrt{x} + c.$$

2 With $u = x$, $v' = e^{-4x}$, so that $u' = 1$, $v = -\frac{1}{4}e^{-4x}$, integration by parts gives:

$$\int_0^{\ln 2} xe^{-4x}dx = \left[-\frac{1}{4}xe^{-4x}\right]_0^{\ln 2} + \frac{1}{4}\int_0^{\ln 2} e^{-4x}dx = -\frac{\ln 2}{4}e^{-4\ln 2} - \frac{1}{4}\left[\frac{1}{4}e^{-4x}\right]_0^{\ln 2} = \frac{15 - 4\ln 2}{256}.$$

3 Use integration by parts with $u = \varphi$, $v' = \csc^2\varphi$:

$$\int \varphi\csc^2\varphi\,d\varphi = -\varphi\cot\varphi + \int \cot\varphi\,d\varphi = -\varphi\cot\varphi + \ln|\sin\varphi| + C.$$

4 Let $u = \tan x$, so $du = \sec^2 x\,dx$:

$$\int \tan^3 x\sec^2 x\,dx = \int u^3\,du = \frac{1}{4}u^4 + c = \frac{1}{4}\tan^4 x + c.$$

5 Set $q = 6\tan\theta$, so $dq = 6\sec^2\theta\,d\theta$. Since $q = 6$ implies $\theta = \pi/4$ and $q = 6\sqrt{3}$ implies $\theta = \pi/3$, we obtain:

$$\int_{\pi/4}^{\pi/3} \frac{36\tan^2\theta \cdot 6\sec^2\theta}{(36\tan^2\theta + 36)^2}d\theta = \int_{\pi/4}^{\pi/3} \frac{\tan^2\theta}{6\sec^2\theta}d\theta = \frac{1}{6}\int_{\pi/4}^{\pi/3}\sin^2\theta\,d\theta = \frac{1}{12}\left(\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}\right).$$

(Note: the formula for $\int \sin^n x\,dx$ on the back of the exam helps.)

6a Making the substitution $3\tan\theta = x$, so $dx = 3\sec^2\theta\,d\theta$, we have

$$\text{Area} = \int_0^4 \frac{1}{\sqrt{x^2+9}}dx = \int_0^{\tan^{-1}(\frac{4}{3})} \sec\theta\,d\theta = \big[\ln|\sec\theta + \tan\theta|\big]_0^{\tan^{-1}(\frac{4}{3})} = \ln 3.$$

6b We have

$$\text{Volume} = \int_0^4 \pi[f(x)]^2dx = \int_0^4 \frac{\pi}{x^2+9}dx = \left[\frac{\pi}{3}\tan^{-1}\frac{x}{3}\right]_0^4 = \frac{\pi}{3}\tan^{-1}\frac{4}{3}.$$

7a First,

$$\frac{12}{(2s-1)(s-6)} = \frac{A}{2s-1} + \frac{B}{s-6} = \frac{-24/11}{2s-1} + \frac{12/11}{s-6},$$

so

$$\int \frac{12}{(2s-1)(s-6)}\,ds = \int\left(\frac{-24/11}{2s-1} + \frac{12/11}{s-6}]\right)ds$$

$$= -\frac{12}{11}\ln|2s-1| + \frac{12}{11}\ln|s-6| + C = \frac{12}{11}\ln\left|\frac{s-6}{2s-1}\right| + C.$$

7b We have

$$\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1},$$

so

$$x - 5 = Ax(x+1) + B(x+1) + Cx^2 = (A+C)x^2 + (A+B)x + B,$$

which yields the system of equations

$$\begin{cases} A + C = 0 \\ A + B = 1 \\ B = -5 \end{cases}$$

The solution to the system is $(A, B, C) = (6, -5, -6)$, so

$$\int \frac{x-5}{x^2(x+1)}\,dx = \int \frac{6}{x}\,dx - \int \frac{5}{x^2}dx - \int \frac{6}{x+1}\,dx = 6\ln|x| + \frac{5}{x} - 6\ln|x+1| + c.$$

8a Let $u = \ln x$, so $du = \frac{1}{x}dx$, and then

$$\int_e^\infty = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_1^{\ln t} \frac{1}{u^2}\,du = \lim_{t\to\infty} \left[-\frac{1}{u}\right]_1^{\ln t} = \lim_{t\to\infty} \left(1 - \frac{1}{\ln t}\right) = 1.$$

8b Using integration by parts en route:

$$\int_0^1 = \lim_{t\to 0^+} \int_t^1 z\ln z\,dz = \lim_{t\to 0^+} \left[z\ln z - z\right]_t^1 = \lim_{t\to 0^+} (t - t\ln t - 1) = -1.$$

9 Let I equal the integral. For $x \in [1, \infty)$ we find that

$$0 \le \frac{1}{\sqrt{x}} \le \frac{1+\sin^2 x}{\sqrt{x}},$$

and since

$$J = \int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/2}dx = \lim_{t\to\infty} \left[2x^{1/2}\right]_1^t = 2\lim_{t\to\infty} (\sqrt{t} - 1) = \infty,$$

so J diverges, the Comparison Theorem implies that I diverges.