Math 141 Exam #1 Key (Fall 2021)

1 By the Inverse Function Theorem $(f^{-1})'(f(a)) = 1/f'(a)$, so apply a bit of trial-and-error to find a such that $f(a) = 2$:

$$f(a) = 2 \quad \Rightarrow \quad a^3 + 3\sin a + 2\cos a = 2 \quad \Rightarrow \quad a = 0.$$

Also

$$f'(x) = 3x^2 + 3\cos x - 2\sin x$$

Then

$$(f^{-1})'(2) = (f^{-1})'(f(0)) = \frac{1}{f'(0)} = \frac{1}{3}.$$

2a $\dfrac{d}{dx}\big[x - \ln(2 + e^x)\big] = 1 - \dfrac{e^x}{2 + e^x}$

2b $\dfrac{d}{dt} e^{2t\ln(1-3t)} = (1 - 3t)^{2t}\left[2\ln(1 - 3t) - \dfrac{6t}{1 - 3t}\right]$

2c $\dfrac{d}{dr}(-r\log_6 r) = -\dfrac{d}{dr}\left(\dfrac{r\ln r}{\ln 6}\right) = -\dfrac{1 + \ln r}{\ln 6}$

2d $\dfrac{2y}{1 + (y^2)^2} + \dfrac{1}{\sqrt{1 - (\sqrt{y})^2}} \cdot \dfrac{1}{2\sqrt{y}} = \dfrac{2y}{1 + y^4} + \dfrac{1}{2\sqrt{y - y^2}}$

2e $6e^{2x}\tanh^2(e^{2x})\,\mathrm{sech}^2(e^{2x})$

3 For $y(x) = 2e^x - 1$ we have $y'(x) = 2e^x$, so $y'(\ln 3) = 6$ is the slope of the line. The line contains point $(\ln 3, y(\ln 3)) = (\ln 3, 5)$, so equation is

$$y = 6x - 6\ln 3 + 5.$$

4a With a long division we have

$$\int_0^3 \frac{2p - 1}{p + 1}\,dp = \int_0^3 \left(2 - \frac{3}{p + 1}\right) dp = \big[2p - 3\ln|p + 1|\big]_0^3 = 6 - 3\ln 4.$$

4b Pull out some constants and then use a given formula:

$$\frac{1}{10}\int \frac{dt}{t\sqrt{t^2 - 2^2}} = \frac{1}{10} \cdot \frac{1}{2}\sec^{-1}\left|\frac{t}{2}\right| + C = \frac{1}{20}\sec^{-1}\left|\frac{t}{2}\right| + C.$$

4c Generally $\int e^{ax}dx = e^{ax}/a$, so,

$$\int 5^{-3x}dx = \int e^{-3x\ln 5}dx = \frac{1}{-3\ln 5}e^{-3x\ln 5} + C = -\frac{5^{-3x}}{3\ln 5} + C.$$

4d Let $u = 3 + 2e^x$, so $e^x\,dx = \frac{1}{2}du$ and the integral becomes

$$\int_5^{11} \frac{1/2}{u}\,du = \frac{1}{2}\big[\ln|u|\big]_5^{11} = \frac{1}{2}\ln\left(\frac{11}{5}\right).$$

5 $f'(x) = 0$ implies $e^{-x^2}(1 - 2x^2) = 0$ implies $x = \pm\frac{1}{\sqrt{2}}$, the critical points for f. Since $f'(-1) < 0$, $f'(0) > 0$, and $f'(1) < 0$, the Intermediate Value Theorem implies $f' < 0$ on $(-\infty, -\frac{1}{\sqrt{2}})$, $f' > 0$ on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $f' < 0$ on $(\frac{1}{\sqrt{2}}, \infty)$. By the First Derivative Test f has a minimum at $x = -\frac{1}{\sqrt{2}}$ and a maximum at $x = \frac{1}{\sqrt{2}}$.

6a With L'Hôpital's Rule,

$$\lim_{x\to 0^+}(3x)^{x/2} = \exp\left[\lim_{x\to 0^+}\frac{\ln(3x)}{2/x}\right] \overset{\text{LR}}{=} \lim_{x\to 0^+}\left(-\frac{x}{2}\right) = e^0 = 1.$$

6b With L'Hôpital's Rule,

$$\begin{aligned}
\lim_{x\to\infty}\exp\left[(2x+1)\ln\left(\frac{2x-3}{2x+5}\right)\right] &= \exp\left[\lim_{x\to\infty}\frac{\ln\left(\dfrac{2x-3}{2x+5}\right)}{\dfrac{1}{2x+1}}\right] \\
&\overset{\text{LR}}{=} \exp\left[\lim_{x\to\infty}\frac{\dfrac{2}{2x-3} - \dfrac{2}{2x+5}}{\dfrac{-2}{(2x+1)^2}}\right] \\
&= \exp\left[-8\lim_{x\to\infty}\frac{4x^2+4x+1}{4x^2+4x-15}\right] \\
&= \exp(-8) = e^{-8}.
\end{aligned}$$