Math 141 Exam #3 Key (Fall 2020)

1a Use L'Hôpital's rule:

$$\lim_{n\to\infty} n\sin\frac{\pi}{n} = \lim_{n\to\infty}\frac{\sin(\pi/n)}{1/n} \overset{\text{LR}}{=} \lim_{n\to\infty}\frac{(-\pi/n^2)\cos(\pi/n)}{-1/n^2} = \lim_{n\to\infty}\pi\cos\frac{\pi}{n} = \pi\cos 0 = \pi.$$

1b We have

$$\lim_{n\to\infty}\left(n-\sqrt{n^2-1}\right) = \lim_{n\to\infty}\frac{(n-\sqrt{n^2-1})(n+\sqrt{n^2-1})}{n+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{1}{n+\sqrt{n^2-1}} = 0.$$

2 The problem is that $\sin n$ converges to no particular value as $n\to\infty$, so we contrive to eliminate it in some fashion. Since $-1\leq \sin n \leq 1$ for all n, it follows that

$$-\frac{3n^2+2n+1}{4n^3+n} \leq a_n \leq \frac{3n^2+2n+1}{4n^3+n}$$

for all $n\geq 1$. Then, observing that

$$\lim_{n\to\infty}\frac{3n^2+2n+1}{4n^3+n} = \lim_{n\to\infty}\left(-\frac{3n^2+2n+1}{4n^3+n}\right) = 0,$$

the Squeeze Theorem implies that $\lim_{n\to\infty} a_n = 0$.

3 Reindex to obtain

$$\sum_{n=1}^{\infty}\frac{6}{4^n} = \sum_{n=0}^{\infty}\frac{6}{4^{n+1}} = \frac{6}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{3}{2}\cdot\frac{1}{1-1/4} = 2.$$

4 The nth partial sum is

$$\begin{aligned} s_n &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + [\ln n - \ln(n-1)] + [\ln(n+1) - \ln n] \\ &= -\ln 1 + \ln(n+1) = \ln(n+1), \end{aligned}$$

and so

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\ln(n+1) = \infty.$$

That is, the series diverges.

5 Find the smallest integer value of n for which $\frac{1}{2n^4} < \frac{1}{1000}$. Since

$$\frac{1}{2n^4} < \frac{1}{1000} \quad\Rightarrow\quad n^4 > 500,$$

and $4^4 < 500$ while $5^4 > 500$, the estimation

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^4} \approx \sum_{n=1}^{4} \frac{(-1)^n}{2n^4} = -\frac{1}{2} + \frac{1}{32} - \frac{1}{162} + \frac{1}{512}$$

has absolute error less than 10^{-3}.

6a For all $n \geq 1$ we have

$$0 < \frac{4}{2+3^n n} \leq \frac{4}{3^n n} \leq \frac{4}{3^n},$$

and since $\sum 4/3^n$ is a convergent geometric series, we conclude by the Direct Comparison Test that the given series converges.

6b Since

$$\lim_{n\to\infty} \frac{4^n}{n^2} = +\infty,$$

the series diverges by the Divergence Test.

6c Use the Limit Comparison Test, comparing with, say, $\sum n^{-7/6}$. Since

$$L = \lim_{n\to\infty} \frac{\frac{\ln n}{n^{4/3}}}{n^{-7/6}} = \lim_{n\to\infty} \frac{\ln n}{n^{1/6}} \overset{\text{LR}}{=} \lim_{n\to\infty} \frac{n^{-1}}{\frac{1}{6}n^{-5/6}} = \lim_{n\to\infty} \frac{6}{\sqrt[6]{n}} = 0,$$

and the series $\sum n^{-7/6}$ is a convergent p-series, the LCT implies that the given series converges also.

6d Since

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!}\right| = 2\lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2\lim_{n\to\infty} \exp\left(n \cdot \ln \frac{n}{n+1}\right) \\ &= 2\exp\left(\lim_{n\to\infty} \frac{\ln n - \ln(n+1)}{1/n}\right) \overset{\text{LR}}{=} 2\exp\left(\frac{1/n - 1/(n+1)}{-1/n^2}\right) \\ &= 2\exp\left(-\lim_{n\to\infty} \frac{n}{n+1}\right) = 2\exp(-1) = \frac{2}{e} < 1,\end{aligned}$$

the series converges by the Ratio Test.

6e Since

$$\lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \exp\left(-\frac{\ln n}{n}\right) = \exp(0) = 1 \neq 0,$$

the series diverges by the Divergence Test.

6f Because the harmonic series $\sum n^{-1}$ is known to diverge and

$$L = \lim_{n\to\infty} \frac{\dfrac{\pi}{2} - \tan^{-1} n}{\dfrac{1}{n}} \overset{\text{LR}}{=} \lim_{n\to\infty} \frac{-\dfrac{1}{n^2+1}}{-\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \in (0,\infty),$$

the Limit Comparison Test implies that the given series also diverges.

7a Since $(1/n^{5/4})$ is a decreasing sequence of nonnegative values such that $1/n^{5/4} \to 0$ as $n \to \infty$, the series converges by the Alternating Series Test. Since $\sum 1/n^{5/4}$ is a convergent p-series, the given series is also absolutely convergent.

7b The series is $\sum a_n$ with $a_n = (-1)^{n+1}/(\sqrt{n}+6)$. Now,

$$L = \lim_{n\to\infty} \frac{|a_n|}{n^{-1/2}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+6} = 1 \in (0,\infty),$$

and since $\sum n^{-1/2}$ is a divergent p-series, the Limit Comparison Test implies that the series $\sum |a_n|$ diverges, and therefore the given series $\sum a_n$ is not absolutely convergent. However, because the sequence

$$b_n = \frac{1}{\sqrt{n}+6}, \quad n \geq 0,$$

is a decreasing sequence of positive numbers such that $b_n \to 0$ as $n \to \infty$, the Alternating Series Test concludes that $\sum a_n$ converges, and therefore $\sum a_n$ is conditionally convergent.