Math 141 Exam #2 Key (Fall 2020)

1a Apply integration by parts with $u = t$ and $v' = \cos 8t$ to get

$$\int t\cos 8t\,dt = \frac{t}{8}\sin 8t - \int \frac{1}{8}\sin 8t\,dt = \frac{t}{8}\sin 8t + \frac{\cos 8t}{64} + C.$$

1b Let I be the integral. Integrate by parts twice, with $u = \ln^2 x$, $v' = x^2$ the first time, and $u = \ln x$, $v' = x^2$ the second time, to get

$$I = \frac{x^3}{3}\ln^2 x - \frac{2}{3}\int x^2 \ln x\,dx = \frac{x^3}{3}\ln^2 x - \frac{2}{3}\left(\frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx\right)$$
$$= \frac{1}{3}x^3\ln^2 x - \frac{2}{9}x^3\ln x + \frac{2}{27}x^3 + C.$$

2a Let $u = \cos 6t$, so integral I becomes

$$I = (1-\cos^2 6t)^2\cos^2 6t\sin 6t\,dt = -\frac{1}{6}\int (1-u^2)^2u^2\,du = -\frac{u^3}{18} + \frac{u^5}{15} - \frac{u^7}{42} + C$$
$$= -\frac{\cos^3 6t}{18} + \frac{\cos^5 6t}{15} - \frac{\cos^7 6t}{42} + C.$$

2b Let $u = \sec 3\theta$, so integral I becomes

$$I = \int (\sec^2 3\theta - 1)\sec^2 3\theta \cdot \tan 3\theta \sec 3\theta\,d\theta = \frac{1}{3}\int (u^2-1)u^2\,du$$
$$= \frac{u^5}{15} - \frac{u^3}{9} + C = \frac{\sec^5 3\theta}{15} - \frac{\sec^3 3\theta}{9} + C.$$

3a Let $x = 2\sin\theta$ to get

$$\int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\cdot 2\cos\theta\,d\theta = \int_0^{\pi/4} 4\sin^2\theta\,d\theta = 4\left(\left[-\frac{\sin\theta\cos\theta}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} d\theta\right) = \frac{\pi-2}{2},$$

using a given reduction of order formula.

3b Let $x = \tan\theta$ to get

$$\int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C = \frac{x}{\sqrt{1+x^2}} + C.$$

4a With partial fraction decomposition integral becomes

$$\int_1^2 \left(\frac{1}{x} + \frac{4}{3x-2}\right)dx = \left[\ln|x| + \frac{4}{3}\ln|3x-2|\right]_1^2 = \frac{11}{3}\ln 2.$$

4b Perform a long division first, then apply partial fraction decomposition:

$$\begin{aligned}\int\left(x-\frac{9x^2-1}{x^3+9x}\right)dx &= \int\left(x+\frac{1/9}{x}-\frac{82x/9}{x^2+9}\right)dx\\ &= \frac{1}{2}x^2+\frac{1}{9}\ln|x|-\frac{82}{9}\int\frac{x}{x^2+9}\,dx\\ &= \frac{1}{2}x^2+\frac{1}{9}\ln|x|-\frac{41}{9}\ln(x^2+9)+C.\end{aligned}$$

5a Let $u = e^x$ first, and then let $u = \tan\theta$:

$$\int\frac{du}{u^2\sqrt{1+u^2}} = \int\frac{\sec\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\,d\theta.$$

Since $\tan\theta = e^x > 0$ we have $\theta \in (0, \pi/2)$, and so $\sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$. With the substitution $w = \sin\theta$, integral becomes

$$\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \int\frac{\cos\theta}{\sin\theta}\,d\theta = \int\frac{1}{w^2}\,dw = -\frac{1}{w}+C = -\frac{\sqrt{1+e^{2x}}}{e^x}+C.$$

5b Apply integration by parts with $u = \tan^{-1}s$, $v' = s^2$:

$$\begin{aligned}\int s^2\tan^{-1}s\,ds &= \frac{s^3}{3}\tan^{-1}s-\frac{1}{3}\int\frac{s^3}{1+s^2}\,ds\\ &= \frac{s^3}{3}\tan^{-1}s-\frac{1}{3}\int\left(s-\frac{s}{1+s^2}\right)ds\\ &= \frac{s^3}{3}\tan^{-1}s-\frac{1}{6}s^2+\frac{1}{6}\ln(1+s^2)+C.\end{aligned}$$

6a Integral diverges:

$$\lim_{t\to-\infty}\int_t^{-1}x^{-1/3}\,dx = \lim_{t\to-\infty}\left[\tfrac{3}{2}x^{2/3}\right]_t^{-1} = \lim_{t\to-\infty}\tfrac{3}{2}(1-t^{2/3}) = -\infty.$$

6b Let $u = t^4$, so integral becomes

$$\lim_{b\to\infty}\int_0^b\frac{t^3}{1+(t^4)^2}\,dt = \lim_{b\to\infty}\int_0^{b^4}\frac{1/4}{1+u^2}\,du = \frac{1}{4}\lim_{b\to\infty}\left[\tan^{-1}u\right]_0^{b^4} = \frac{\pi}{8}.$$

6c The integral $\int_1^{10}$ we break into $\int_1^2+\int_2^{10}$. Now,

$$\int_1^2 = \lim_{t\to2^-}\int_1^t\frac{dx}{(x-2)^{4/3}} = \lim_{t\to2^-}\left[-3(x-2)^{-1/3}\right]_1^t = -3\lim_{t\to2^-}\left[\frac{1}{\sqrt[3]{t-2}}+1\right] = \infty,$$

so $\int_1^2$ diverges, and therefore $\int_1^{10}$ likewise diverges.

7 Volume is

$$\int_1^\infty \pi(x^{-2})^2\,dx = \pi \lim_{t\to\infty} \int_1^t x^{-4}\,dx = -\frac{\pi}{3} \lim_{t\to\infty} \left(\frac{1}{t^3} - 1\right) = \frac{\pi}{3}.$$

8 Let I be the given integral. For $x \geq 3$ we have

$$\frac{1}{\sqrt{x^7-1}} > \frac{1}{\sqrt{x^7}} \quad \Rightarrow \quad \frac{x^3}{\sqrt{x^7-1}} > \frac{x^3}{\sqrt{x^7}} \quad \Rightarrow \quad \frac{x^3}{\sqrt{x^7-1}} > \frac{1}{\sqrt{x}},$$

and since

$$\int_3^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \left[2\sqrt{x}\right]_3^t = \lim_{t\to\infty} \left(2\sqrt{t} - 2\sqrt{3}\right) = \infty,$$

the Comparison Theorem implies that I diverges.